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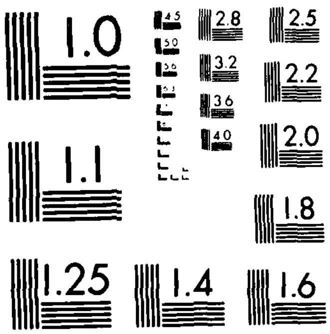

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Chapel Hill, North Carolina

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Limit behaviour for stochastic monotonicity and applications

by

Harry Cohn

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Limit behaviour for stochastic monotonicity and applications

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Abstract

A transition probability function P is said to be stochastically monotone if $P(x,(-\infty,y])$ is non-increasing in x for every fixed y. A (non-homogeneous) Markov chain or process is said to be stochastically monotone if its transition probability functions are stochastically monotone. Diffusions, random walks, birth-and-death and branching processes are examples of such models. It is shown that stochastically monotone processes exhibit two basic types of asymptotic behaviour. Chains with stationary transition probabilities display a cyclic pattern, and a suitably normed and centered chain turns out to converge almost surely if it is geometrically growing. Applications to diffusions and branching processes are added.

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1. Introduction and Summary. We shall start off by considering two examples of stochastically monotone (SM) sequences exhibiting rather contrasting sample path behaviour. Let $\{\xi_n\}$ be a sequence of i.i.d. random variables with mean 0 and variance 1, and $S_n = \xi_1 + \ldots \xi_n$. It is well-known that $\{S_n/\sqrt{n}\}$ converges in distribution to the standard normal distribution N(0,1) and $P(\liminf_{n\to\infty} S_n/\sqrt{n} = -\infty)$ $= P(\limsup_{n\to\infty} S_n/\sqrt{n} = \infty) = 1$. Consider further a supercritical Galton-Watson process $\{Z_n\}$ defined as $Z_{n+1} = \sum_{i=1}^{Z_n} \xi_{n,i}$ where $\{\xi_{n,i}\}$ are i.i.d. conditional on Z_n and $P(\xi_{n,i} = k) = p_k$, $k = 0,1,\ldots$ If $m = \sum_{k=0}^{\infty} kp_k \in (1,\infty)$ it is known (see e.g. [3]) that there exist some norming constants $\{c_n\}$ with $\lim_{n\to\infty} c_{n+1}/c_n = m$ such that $\{Z_n/c_n\}$ converges a.s. to a random variable W whose distribution function is continuous and strictly increasing on $(0,\infty)$. Both cases are instances of SM Markov chains with stationary transition probabilities $\{X_n\}$ for which there exist norming constants $\{a_n\}$ such that $\{a_nX_n\}$ converges in distribution to a non-degenerate limit F. We shall see that under rather general conditions, the growth rate of the norming constants $\{a_n\}$ determines the limit pattern of $\{a_nX_n\}$ and characterizes its limit distribution: if $\lim_{n\to\infty} a_{n+1}/a_n = 1$ then $P(\liminf_{n\to\infty} a_nX_n \le \inf \operatorname{supp} F) = P(\limsup_{n\to\infty} a_nX_n \ge \sup \operatorname{supp} F) = 1$, whereas if $\lim_{n\to\infty} a_{n+1}/a_n \neq 1$ then $\{a_nX_n\}$ converges a.s., the supp F is either the real line or one of its half-lines, F is strictly increasing on supp F, and continuous except maybe for $x = 0$.

A transition probability function P is said to be SM if $P(x,(-\infty,y])$ is non-increasing in x for every fixed y. A non-homogeneous Markov process

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$\{X(t);\ t \in [0,\infty)\}$ (or chain $\{X_n\colon n \geq 0\}$) is said to be SM if its transition probability functions are SM. If $\{X_n\}$ is a discrete time non-homogeneous Markov chain, the stochastic monotonicity of the one-step transition probabilities $\{P_n\}$ suffices for the stochastic monotonicity of $\{X_n\}$ (see [12] Theorem 1). The term "stochastic monotonicity" was coined in [12] and has made the object of intensive study in a number of articles and monographs (see [10], [14], [15], [17] and [26]). Two recent papers ([7] and [2]) have dealt with SM from the point of view of the limit behaviour. In [7] criteria for convergence is probability or a.s. convergence have been derived for chains converging in distribution to non-degenerate limits. In the case when F is continuous such criteria were shown to be necessary and sufficient. The object of investigation in [2] was the self-normalized process $\{F_n(X_n)\}$, where F_n is the distribution function of X_n, under the assumption

$$\sup_x P(X_n = x) \to 0 \tag{1.1}$$

Under (1.1), $\{F_n(X_n)\}$ converges in distribution to the uniform distribution on [0,1]. Among other properties of interest, [2] contains a detailed description of the case when a.s. convergence fails. It turns out that the sample space Ω can be partitioned into some sets $\Omega_1, \Omega_2, \ldots$ and $\Omega_1', \Omega_2', \ldots$ If $W_n = F_n(X_n)$ then for $\omega \in \Omega_i$ $\lim_{n\to\infty} W_n(\omega)$ exists, whereas if $\omega \in \Omega_i'$ then there exist two numbers a_i and b_i with $a_i < b_i$ such that $\liminf_{n\to\infty} W_n(\omega) = a_i$ and $\limsup_{n\to\infty} W_n(\omega) = b_i$. A pictorial description of this sample path behaviour is given in Fig. 1.1 below.

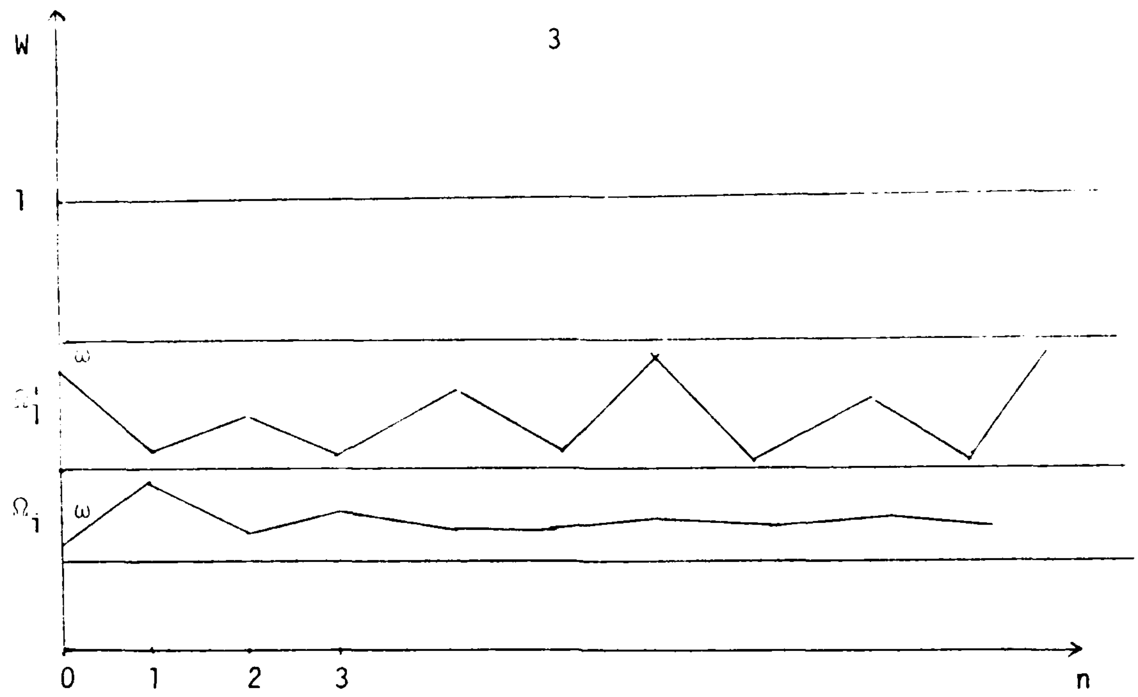

Figure 1.1

When $\{W_n\}$ converges a.s. the sets $\{\Omega_i'\}$ are absent, whereas in the situation described by the example of $\{S_n/\sqrt{n}\}$, $\{\Omega_i\}$ and all but one of $\{\Omega_i'\}$ are absent.

If $\{X_n\}$ converges in distribution to a limit F admitting jump points, convergence a.s. or otherwise may occur irrespective of the properties of $\{W_n\}$. Indeed, the strong law of large numbers for $\{S_n/n\}$ is not prevented by $P(\liminf_{n\to\infty} W_n = 0) = P(\limsup_{n\to\infty} W_n = 1) = 1$. It would be therefore of interest to study the limit behaviour of $\{X_n\}$ when its limit distribution is not necessarily a smooth one and even when convergence in distribution does not hold. We shall give here a new approach to SM processes which does not require convergence in distribution or condition (1.1) and enables one to study several aspects of the limit behaviour. Our method is based on the existence of some random variables $\{W_q\}$ such that $E(W_q|X_n) = \lim_{k\to\infty} P(X_{n_k} \in J_{n_k}|X_n)$ a.s. for any subsequence $\{n_k\}$ and left-unbounded intervals $\{J_{n_k}\}$ with $\lim_{k\to\infty} P(X_{n_k} \in J_{n_k}) = q$,

$q \in (0,1)$ and $n = 0,1,\ldots$ The variables $\{W_q\}$ may admit at most three different values with positive probability. If $P(W_q = 0) = 1-P(W_q = 1)$ W_q will be said to be of type I, and of type II otherwise. Type I variables are of the kind studied in [7] in connection with almost sure convergence, whereas type II variables characterize some features similar to those descrited in [2] for the case when almost sure convergence fails. In Section 2, in addition to describing $\{W_q\}$, we give some results relating limit properties of subsequences of $\{X_n\}$ as well as a simple criterion for a sequence to be mixing. In Section 3 we study sequences converging in distribution where we shall find it convenient to introduce two types of limit points and characterize the limit behaviour in each case. The object of Section 4 is the limit behaviour of suitably normed and centered Markov chains and processes with stationary transition probabilities. In Section 5 criteria for a.s. convergence are derived under some assumption of tightness. Finally, Section 6 contains some applications to branching processes and diffusions.

The main ingredient of the approach is the identification of the sequence of conditional limit distributions as a martingale, which leads to the limit variables $\{W_q\}$. A reader interested in the applications of Section 6 may skip most of the sections and choose to read only part of Section 2 including Theorem 2.1 and the results referrred to in the arguments of Section 6.

2. The general case. Let $S \subseteq R$ be the state-space of $\{X_n\}$, $(\Omega, \mathcal{F}, P)$ its underlying probability space with $\Omega = S \times S \times \ldots$, and $\omega = (\omega_0, \omega_1, \ldots, \omega_n, \ldots)$ the generic element of Ω. We shall agree to write $A = B$ a.s. if $1_A = 1_B$ a.s. and $\lim_{n \to \infty} 1_{A_n} = 1_A$ a.s., 1 being the indicator function of a set. Similarly, we shall say that $\{A_n\}$ converges a.s. if $\{1_{A_n}\}$ does so. Further $A \setminus B$ denotes the difference of the sets A and B and $A \Delta B$ is the symmetric difference of A and B.

We shall say that $\{X_n\}$ converges weakly to a limit F if $\lim_{n\to\infty} F_n(x) = F(x)$ for any continuity point x of F, the case $F(\infty) < 1$ and/or $F(-\infty) > 0$ being not excluded. If $F(-\infty) = 0$ and $F(\infty) = 1$ we shall say that $\{X_n\}$ converges in distribution to F.

Let q be a number with $0 < q < 1$ and assume that there exist a subsequence $\{n_k\}$ of the non-negative integers and some intervals $\{J_{n_k}\}$ such that $\lim_{k\to\infty} P(X_{n_k} \in J_{n_k}) = q$, where $J_{n_k} = (-\infty, x_{n_k}]$ or $(-\infty, x_{n_k})$ for some $\{x_{n_k}\}$. Consider further the quantities $\{P(X_{n_k} \in J_{n_k} | X_n = x)\}$ for $x \in \text{supp } F_n$ and $n > n_k$. By stochastic monotonicity $P(X_{n_k} \in J_{n_k} | X_n = x)$ is non-increasing in x, and by the well-known weak compactness principle (see e.g. [22] p. 181) one can extract a subsequence of $\{n_k\}$, say $\{n_k'\}$, *such that* $G_x^{(n)}(q) = \lim_{k\to\infty} P(X_{n_k'} \in J_{n_k'} | X_n = x)$ exists for all $x \in \text{supp } F_n$. The process of extracting subsequences may be carried out by the well-known diagonal procedure to produce a subsequence $\{n_k^*\}$ of $\{n_k'\}$ such that $G_x^{(n)}(q) = \lim_{k\to\infty} P(X_{n_k^*} \in J_{n_k^*} | X_n = x)$ exists for all $x \in \text{supp } F_n$ and $n = 0,1,\ldots$

<u>Lemma 2.1</u>. There exists a random variable W_q such that $\lim_{n\to\infty} G_{X_n}^{(n)}(q) = W_q$ a.s., $E(W_q) = q$ and $E(W_q | X_n) = G_{X_n}^{(n)}(q)$ a.s. for $n = 0,1,\ldots$

<u>Proof</u>. Applying the Chapman-Kolmogorov formula to $P(X_{n_k^*} \in J_{n_k^*} | X_n = x)$ and then taking the limit as $k\to\infty$ yields

$$G_x^{(n)}(q) = \int G_y^{(n+1)}(q)\, P_{n+1}(x,dy) \qquad n=0,1,\ldots \tag{2.1}$$

It is easy to check that the property (2.1) defining a so-called space-time harmonic function $G_x^{(n)}(q)$ leads to

$$(2.2) \qquad E(G_{X_{n+1}}^{(n+1)}(q) \mid X_n) = G_{X_n}^{(n)}(q) \quad \text{a.s.}$$

The Markov property in conjunction with (2.2) implies that $\{G_{X_n}^{(n)}(q)\}$ is a martingale. Because $\{G_x^{(n)}(q)\}$ are bounded, $\lim_{n\to\infty} G_{X_n}^{(n)}(q) = W_q$ a.s. exists. The total probability formula yields $E(G_{X_n}^{(n)}(q)) = E(W_q) = q$ and by the closure property for bounded martingales we conclude that $E(W_q \mid X_n) = G_{X_n}^{(n)}(q)$ a.s. for $n=0,1,2,\ldots$

Remark 2.1. The functions $\{G_x^{(n)}(q)\}$ and therefore the limit variables $\{W_q\}$ seem to depend on the choice of the subsequence $\{n_k^*\}$ extracted from $\{n_k\}$ at this stage in the proof. It will turn out that $G_x^{(n)}(q)$ are independent of the choice of $\{n_k^*\}$ and even that $G_x^{(n)}(q) = \lim_{k\to\infty} P(X_{n_k} \leq x_{n_k} \mid X_n = x)$ whenever $\lim_{k\to\infty} P(X_{n_k} \leq x_{n_k}) = q$ for any $\{n_k\}$ (which may even be the set of non-negative integers). The variables $\{W_q\}$ will turn out to characterize the limit behaviour of $\{X_n\}$.

Proposition 2.1. Let z be a continuity point of the distribution function of W_q. Then

$$(a) \qquad \lim_{n\to\infty} \{X_n \in I_n\} = \{W_q < z\} \quad \text{a.s.}$$

where $I_n = (-\infty, z_n)$ or $(-\infty, z_n]$ for some real numbers $\{z_n\}$.

(b) either $\{W_q = 1\} = \{W_q > z\}$ a.s. or $\{W_q = 0\} = \{W_q < z\}$ a.s.

<u>Proof</u>. According to Lemma 2.1 $\{G^{(n)}_{X_n}(q)\}$ converges a.s. to W_q. This implies that for any continuity point z of F

$$(2.3) \qquad \{W_q \leq z\} = \lim_{n\to\infty} \{G^{(n)}_{X_n}(q) \leq z\} \quad \text{a.s.}$$

Stochastic monotonicity and (2.3) ensure the existence of some left-unbounded intervals $\{I_n\}$ such that $\lim_{n\to\infty} \{X_n \in I_n\} = \{W_q > z\}$ a.s. and (a) is proved.

To prove (b) notice first that two cases may arise: (i) $P(W_q > z) \leq q$ or (ii) $P(W_q > z) > q$. We assume (i) and claim that $G^{(n)}_{X_n}(q) \geq P(W_q > z|X_n)$ a.s. Indeed, by the proven part (a) above $P(W_q > z|X_n) = \lim_{m\to\infty} P(X_m \in I_m|X_n)$ a.s. Assume by way of contradiction that $G^{(n)}_{X_n}(q) < P(W_q > z|X_n)$ on a set A of positive probability. Since $\{I_n\}$ are left-unbounded intervals, this may happen only if $z_{n^*_k} > x_{n^*_k}$ for k large enough. It follows that $G^{(n)}_{X_n}(q) \geq P(W_q > z|X_n)$ a.s. with strict inequality on A. Taking expectations gives $q > P(W_q > z)$, a contradiction that proves that $G^{(n)}_{X_n}(q) \geq P(W_q > z|X_n)$ a.s. Combining the latter inequality with the Markov property and the martingale convergence theorem yields $W_q = 1$ for almost all $\omega \in \{W_q > z\}$. If (ii) holds one gets $G^{(n)}_{X_n}(q) < P(W_q > z|X_n)$ a.s. and a similar reasoning leads to $W_q = 0$ for almost all $\omega \in \{W_q < z\}$.

Proposition 2.1(b) shows that W_q may take at most three distinct values with positive probability, two of them being 0 and 1. We shall say that W_q is of type I if $P(W_q = 0) = 1-P(W_q = 1)$, and of type II otherwise. For W_q of type II the possible values are 0, k_q and 1 where $0 < k_q < 1$. Clearly $E(W_q) = q$ implies $\min(P(W_q = 0), P(W_q = 1)) > 0$ for $q \in (0,1)$ in case I, whereas in case II $P(W_q = 0) = 0$ and/or $P(W_q = 1) = 0$ is a possibility, but $P(W_q = k_q) > 0$ holds anyway.

Lemma 2.2 (a) If W_q is of type I then there exists a sequence of left-unbounded intervals $\{I_n\}$ such that $\lim_{n\to\infty} \{X_n \in I_n\} = \{W_q = 1\}$ a.s. and $\lim_{n\to\infty} P(X_n \in I_n) = q$.

(b) If there exists a subsequence $\{k_n\}$ of positive integers and some left-unbounded intervals $\{I_{k_n}\}$ such that $\lim_{n\to\infty} \{X_{k_n} \in I_{k_n}\}$ a.s. exists, then W_q is of type I with $q = \lim_{n\to\infty} (X_{n_k} \in I_{k_n})$.

Proof If W_q is of type I then $P(W_q = 1) = q$ and $\{W_q > z\} = \{W_q = 1\}$ for any z with $0 < z < 1$, in which case Proposition 2.1 (a) implies that $\lim_{n\to\infty} \{X_n \in I_n\} = \{W_q = 1\}$ a.s. for some left-unbounded intervals $\{I_n\}$. This necessarily entails $\lim_{n\to\infty} P(X_n \in I_n) = P(W_q = 1) = q$, and (a) is proved.

Assume now that condition (b) is in force and notice that $P(A|X_m) = \lim_{n\to\infty} P(X_{k_n} \in I_{k_n}|X_m)$ a.s. where $A = \lim_{n\to\infty} \{X_{k_n} \in I_{k_n}\}$ a.s. Further, as in the proof of Proposition 2.1 (a), one can invoke the martingale convergence theorem and stochastic monotonicity to deduce that $\lim_{m\to\infty} P(A|X_m) = 1_A$ a.s. implies the existence of some left-unbounded intervals $\{I_n\}$ such that $\lim_{n\to\infty} \{X_n \in I_n\} = A$ a.s. To complete the proof we shall show that $1_A = W_q$ a.s. where $W_q = \lim_{n\to\infty} G^{(n)}_{X_n}(q)$ a.s. and $q = P(A)$. Indeed, recall that

$G^{(m)}_{X_m}(q) = E(W_q|X_m) = \lim_{k\to\infty} P(X_{n_k^\star} \in J_{n_k^\star}|X_m)$ a.s. where $\lim_{k\to\infty} P(X_{n_k^\star} \in J_{n_k^\star}) = q$.

Since $\{I_n\}$ are left-unbounded intervals, it is easy to see that $\lim_{k\to\infty} P(\{X_{n_k^\star} \in J_{n_k^\star}\} \triangle \{X_{n_k^\star} \in I_{n_k^\star}\}) = 0$. Thus, if necessary by taking a further subsequence, one can arrange to have $\sum_{k=1}^{\infty} P(\{X_{n_k^\star} \in J_{n_k^\star}\} \triangle \{X_{n_k^\star} \in I_{n_k^\star}\}) < \infty$. By the Borel-Cantelli lemma $P(\{X_{n_k^\star} \in J_{n_k^\star}\} \neq \{X_{n_k^\star} \in I_{n_k^\star}\}$ i.o.$) = 0$, and this yields $A = \lim_{k\to\infty} \{X_{n_k^\star} \in J_{n_k^\star}\}$ a.s. Thus $E(W_q|X_n) = P(A|X_n)$ a.s. for all n, which implies $W_q = 1_A$ a.s. and proves (b).

Lemma 2.3. Suppose that W_q is of type II and write $q_1 = P(W_q = 1)$ and $q_2 = 1 - P(W_q = 0)$. Then

(a) *If* $P(W_q = 1) > 0$ and/or $P(W_q = 0) > 0$, *then* W_{q_1} and/or W_{q_2} *are of type I,* $\{W_q = 1\} = \{W_{q_1} = 1\}$ *a.s. and* $\{W_q > 0\} = \{W_{q_2} = 1\}$ *a.s.*

(b) *There is no* q' *in* (q_1, q_2) *with* $W_{q'}$ *of type I.*

Proof. According to Proposition 2.1(a) for z such that $k_q < z < 1$ in case that $P(W_q = 1) > 0$ or $0 < z < k_q$ in case that $P(W_q = 0) > 0$, we get that there exist some left-unbounded intervals $\{I'_n\}$ or $\{I''_n\}$ with $\lim_{n\to\infty} \{X_n \in I'_n\} = \{W_q = 1\}$ a.s. or $\lim_{n\to\infty} \{X_n \in I''_n\} = \{W_q > 0\}$ a.s. respectively, and Lemma 2.2(b) completes the proof of (a).

To prove (b) assume the contrary and choose $q' \in (q_1, q_2)$ such that W'_q is of type I. Then two cases may occur: (i) $q < q'$ or (ii) $q > q'$. Since W_q is monotone in q, (i) implies $1-q_2 = P(W_q = 0) \geq P(W_{q'} = 0) = 1-q'$ contradicting $q_2 > q'$, whereas if (ii) holds $q_1 = P(W_q = 1) \geq P(W_{q'} = 1) = q'$ contradicting $q_1 < q'$ and completing the proof.

A set A with $A = \lim_{n\to\infty} \{X_n \in J_n\}$ a.s. for some intervals $\{J_n\}$ is said to be for $x \in A_n$ and n large enough. Thus (3.5) holds and $\{X_n \in B_n \text{ i.o.}\} = \Lambda(a,b)$ a.s. Since $F(b-)-F(b-\varepsilon) > 0$ for any $\varepsilon > 0$, ε_1 and ε_2 may be chosen arbitrarily close to b and therefore $\limsup_{n\to\infty} X_n = b$ for almost all $\omega \in \Lambda(a,b)$.

It remains to consider the case $F(b-) < q_2$ which makes b a point of type I'. Clearly $F(b) \geq q_2$ and $F(b) - F(b-) > 0$. This is similar to the case of a considered before and may be dealt with by taking $A_n = \{x:F_x^{(n)}(b)>k(b)+\varepsilon\} \cap \{x:k(b-) - \varepsilon \leq F_x^{(n)}(b-) \leq k(b-) + \varepsilon\}$ or $A_n = \{x: F_x^{(n)}(b) > k(b-) + 2\varepsilon\} \cap \{x:k(b-) - \varepsilon \leq F_x^{(n)}(b-) \leq k(b-) + \varepsilon\}$ according as b is of type II_1 or I, $0 < k(b-)-\varepsilon < k(b-) + 2\varepsilon < 1$ and $B_n = (b - \varepsilon_n, b + \varepsilon_n)$ for some positive $\{\varepsilon_n\}$ with $\lim_{n\to\infty} \varepsilon_n = 0$ and $\lim_{n\to\infty} P(X_n \in B_n) = F(b) - F(b-)$.

Consider now the case s_1=a and/or s_2=b. Since there are no points of type I' smaller or equal to a and/or larger or equal to b, the above proof may be easily modified to yield $\liminf_{n\to\infty} X_n \leq a$ and/or $\limsup_{n\to\infty} X_n \geq b$ for almost all $\omega \in \Lambda(a,b)$. We recall that the Borel-Cantelli lemma makes it possible that $P(X_n = x_n \text{ i.o.}) > 0$ for a sequence $\{x_n\}$ with $\lim_{n\to\infty} P(X_n = x_n) = 0$. If such a sequence with $\lim_{n\to\infty} x_n = s_1$ exists, then $a^* = a$, and similarly if such a sequence with $\lim_{n\to\infty} x_n = s_2$ exists then $b^* = b$ and the proof is finished.

Under some restrictions on F, Theorems 3.1(a) and 3.2(a) were derived in [7]. If F is the uniform distribution on [0,1], Theorems 3.1(b) and 3.2(b) may be extracted from Proposition (3.1) of [2].

Remark 3.2. Conditions of the type $G_x(y) = 1$ or 0 according as $x < y$ or $x \geq y$ were considered in [7] in relation to a.s. convergence and proved

Then Theorem 9.5.2 of [4] applies and yields $P(\{X_n \in A_n \text{ i.o.}\} \setminus \{X_n \in B_n \text{ i.o.}\}) = 0$. Notice that

(3.6) $$P\Big(\bigcup_{j=n+1}^{\infty} \{X_n \in B_j\} \,|\, X_n = x\Big) \geq \lim_{j\to\infty} P(\{X_j \in B_j\}|X_n=x)$$

$$= F_x^{(n)}(a) - F_x^{(n)}(a-)$$

Taking into account the definition of $\{A_n\}$ we get $F_x^{(n)}(a) - F_x^{(n)}(a-) \geq k(a) - \varepsilon - \eta > 0$ for $x \in A_n$ and n sufficiently large, which proves (3.5) for $\delta = k(a) - \varepsilon - \eta$. It follows that $\liminf_{n\to\infty} X_n = a$ for almost all $\omega \in \Lambda(a,b)$.

We prove now that $\limsup_{n\to\infty} X_n = b$ for almost all $\omega \in \Lambda(a,b)$. Assume first that $F(b-) = \alpha_2$ in which case we may choose ε_1 and ε_2 with $\varepsilon_1 > \varepsilon_2 > 0$ such that $b - \varepsilon_1$ and $b - \varepsilon_2$ are continuity points of F, $(b - \varepsilon_1, b - \varepsilon_2) \subset I$ and $F(b - \varepsilon_2) - F(b - \varepsilon_1) > 0$. Define $A_n = \{x: k(b-\varepsilon_1) - \varepsilon \leq F_x^{(n)}(b - \varepsilon_1) \leq k(b - \varepsilon_1) + \varepsilon\} \cap \{x: k(b - \varepsilon_2) - \varepsilon \leq F_x^{(n)}(b - \varepsilon_2) \leq k(b - \varepsilon_2) + \varepsilon\}$ where ε may be chosen such that $k(b - \varepsilon_1) - \varepsilon$, $k(b - \varepsilon_1) + \varepsilon$, $k(b - \varepsilon_2) - \varepsilon$ and $k(b - \varepsilon_2) + \varepsilon \in (0,1)$ and $k(b - \varepsilon_2) - k(b - \varepsilon_1) - 2\varepsilon > 0$. Take $B_n = (b - \varepsilon_1, b + \varepsilon_2)$. Then again we get $\lim_{n\to\infty} \{X_n \in A_n\} = \Lambda(a,b)$ a.s. and we shall show that (3.5) obtains in this case as well. Indeed

(3.7) $$P\Big(\bigcup_{j=n+1}^{\infty} \{X_j \in B_j\} \,|\, X_n = x\Big) \geq \lim_{n\to\infty} P(\{X_j \in B_j\}|X_n=x)$$

$$= F_x^{(n)}(b - \varepsilon_2) - F_x^{(n)}(b - \varepsilon_1)$$

$k(b -$ [illegible] $- k($ [illegible] $-$

Theorem 3.2. Suppose that $\{X_n\}$ is a SM Markov chain converging in distribution to F. Then for any $y \in \Gamma$ one of the following two cases occurs:

(a) y is of type I, in which case there exist some numbers $\{y_n\}$ and intervals $\{I_n\}$, where I_n is either $(-\infty, y_n)$ or $(-\infty, y_n]$, such that $\lim_{n\to\infty} \{X_n \in I_n\}$ a.s. exists and $\lim_{n\to\infty} P(X_n \in I_n) = F(y)$.

(b) y is of type II, in which case there exist an interval I containing y with end-points a and b, $a < b$ and an event $\Lambda(a,b)$ with $P(\Lambda(a,b)) > 0$ such that $\liminf_{n\to\infty} X_n = a^*$ and $\limsup_{n\to\infty} X_n = b^*$ for almost all $\omega \in \Lambda(a,b)$ and some constants a^* and b^*. In addition, $a^* \leq a$ or $= a$ according as $a = s_1$ or $> s_1$, and $b^* \geq b$ or $= b$ according as $b = s_2$ or $< s_2$.

Proof. (a) follows from Lemma 2.2(a). It is clear that $\{y_n\}$ may not converge to y if the set $\{x : u = F(x)\}$ with $u = F(y)$ has more than one point.

To prove (b) assume first that $a > s_1$ and $b < s_2$. Set for definiteness $I = [a,b)$ where a and b are finite. The case $I = (a,b)$ is simpler, since then a may be treated like b with $F(b-) = q_2$, a case that will be taken up further on. Thus, assume a to be to of type II, and write $A_n = \{x : k(a) - \varepsilon \leq F_x^{(n)}(a) \leq k(a) + \varepsilon\} \cap \{x : F_x^{(n)}(a-) < \eta\}$ where $0 < k(a) - \varepsilon < k(a) + \varepsilon < 1$ and $0 < \eta < k(a) - \varepsilon$. Since $\lim_{n\to\infty} F_{X_n}^{(n)}(a) = k(a)$ for almost all ω ($\lim_{n\to\infty} F_{X_n}^{(n)}(a-) \leq \lim_{n\to\infty} P(W(a) = 1 | X_n) = 0$) we get that $\lim_{n\to\infty} \{X_n \in A_n\} = \Lambda(a,b)$ a.s. Write further $B_n = (a - \varepsilon_n, a + \varepsilon_n)$ where $\{\varepsilon_n\}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$ and $\lim_{n\to\infty} P(X_n \in B_n) = F(a) - F(a-)$. We shall show that for some δ with $0 < \delta < 1$ and n large enough

$$\text{(3.5)} \qquad P\Big(\bigcup_{j=n+1}^{\infty} \{X_j \in B_j\} \,\Big|\, X_n\Big) \geq \delta \quad \text{for almost all } \omega \in \{X_n \in A_n\}$$

$\lim_{n\to\infty} P(X_n \in I_n^0) = F(y)$. It is easy to see that $F(x^0) < F(y)$ and therefore $x_n^0 < y_n^0$ for n sufficiently large, which together with stochastic monotonicity yields $F^{(n)}_{x_n^0}(y) \geq F^{(n)}_{y_n^0}(y)$ where $F^{(n)}_{y_n^0} = \lim_{x \uparrow y_n^0} F_x^{(n)}(y)$. However, $F^{(n)}_{y_n^0}(y) \geq z_0$ and $\lim_{n\to\infty} F^{(n)}_{x_n^0}(y) < z_0$ is contradicted. Thus $\lim_{n\to\infty} F^{(n)}_{x_n}(y) = 1$ whenever $\lim_{n\to\infty} x_n = x < x_0(y)$. The proof for $x > x_1(y)$ may be derived by a similar reasoning.

Consider now the case when y is of type II. We shall prove that $\lim_{n\to\infty} F^{(n)}_{x_n}(y) = k(y)$ for any $\{x_n\}$ with $q_1 < \lim_{n\to\infty} F_n(x_n) < q_2$. Suppose that the contrary holds and take for definiteness $\lim_{n\to\infty} F^{(n)}_{x_n'}(y) < k(y)$ for some $\{x_n'\}$ such that $q_1 < \lim_{n\to\infty} F_n(x_n') = q < q_2$. By stochastic monotonicity $\lim_{n\to\infty} F^{(n)}_{x_n}(y) < k(y)$ wherever $x_n \leq x_n'$, and since $\{F^{(n)}_{x_n}(y)\}$ converges a.s. to $W(y)$ as $n \to\infty$ we get $P(W(y) < k(y)) = P(W(y) = 0) \geq 1-q > 1-q_2$, which is impossible. Since the case $x < a$ or $x > b$ may be dealth with as in the proof given above for y of type I, it remains to notice that $k(y) = (F(y) - q_1)/P(\Lambda(a,b))$ follows from the more general result of Theorem 2.1(b).

Remark 3.1. If y is of type II, there must exist at least two points x and sequences $\{x_n\} \subset \{\text{supp } F_n\}$ with $\lim_{n\to\infty} x_n = x$ and $q_1 < \lim_{n\to\infty} F_n(x_n) < q_2$. Indeed, $q_1 < F(y) < q_2$ guarantees (3.4) for some sequences $\{x_n\} \subset \{\text{supp } F_n\}$ with $\lim_{n\to\infty} F_n(x_n)$ in each of $(q_1, F(y))$ and $(F(y), q_2)$.

Write $s_1 = \inf\{x: x \in \text{supp } F\}$ and $s_2 = \sup\{x: x \in \text{supp } F\}$.

there exists a subsequence $\{x_{n_k}\}$ with $x_{n_k} \in \text{supp } F_{n_k}$ and $\lim_{k\to\infty} x_{n_k} = x$ but not a whole sequence $\{x_n\}$, case that could not happen if $x \in \text{supp } F$. In what follows we shall write for convenience $\lim_{n\to\infty} x_n = x$ wherever $x \in U$, a relation that should be understood to be replaced by $\lim_{k\to\infty} x_{n_k} = x$ when no such $\{x_n\}$ with $x_n \in \text{supp } F_n$ exists, the arguments used in the proofs being the same. Write $x_0(y) = \inf \{x: F(x) = F(y)\}$ and $x_1(y) = \sup \{x: F(x) = F(y)\}$.

Theorem 3.1. Suppose that $\{X_n\}$ is a SM Markov chain converging in distribution to F. Then

$$\lim_{n\to\infty} F_{x_n}^{(n)}(y) = G_x(y) \tag{3.4}$$

holds for

(a) any $y \in \Gamma$, $x \in U$ and $\{x_n\} \in \{\text{supp } F_n\}$ with $\lim_{n\to\infty} x_n = x$, except maybe for $x \in [x_0(y), x_1(y)]$, for y of type I and

(b) any $y \in \Gamma \setminus \{a,b\}$, $x \in U$ and $\{x_n\} \in \{\text{supp } F_n\}$ with $\lim_{n\to\infty} x_n = x$ such that $\lim_{n\to\infty} F_n(x_n)$ exists and differs from q_1 and q_2, for y of type II.

Proof. Consider first the case when y is of type I and assume by way of contradiction that there exist $x^0 < x_0(y)$ and z_0 with $0 < z_0 < 1$ such that $\lim_{n\to\infty} F_{x_n^0}^{(n)}(y) \neq z_0$ for a sequence $\{x_n^0\}$ with $\lim_{n\to\infty} x_n^0 = x^0$ and $x^0 \in U$. We may suppose without loss of generality that z_0 is a continuity point of F and get as in the proof Proposition 2.1(a) that if $I_n^0 = \{x: F_x^{(n)}(y) > z_0\}$ then $\lim_{n\to\infty} \{X_n \in I_n^0\} = \{W(y) = 1\}$ a.s., where I_n^0 is either $(-\infty, y_n^0]$ or $(-\infty, y_n^0)$ and

b is of type II write $q_1' = P(W(b) = 1)$ and $q_2' = P(W(b) \geq k(b))$ and notice that $F(b) = E(W(b)) > q_2 > E(W(y)) = F(y)$ requires $(q_1,q_2) \neq (q_1',q_2')$ which in conjunction with Lemma 2.3(b) leads to $q_2 = q_1'$. It follows that $F(b-) \leq q_1' < F(b)$ which makes b a point of type II_1. On the other hand, $\{W(b) = k(b)\} \neq \{W(y) = k(y)\}$ a.s. and $b \notin I$ obtains in either case.

To complete the proof notice that by Theorem 2.1(b) $\Lambda_{q_1,q_2} = \{W(y) = k(y)\}$ a.s. for any y with $q_1 < F(y) < q_2$, i.e. for any y in I.

We shall next introduce two types of limit distributions $\{G_x(y)\}$ corresponding to the types of y defined above. Let $\Gamma = \{x: 0 < F(x) < 1\} \cap C(F)$ where $C(F)$ is the set of continuity points of F. Suppose that $y \in \Gamma$ is of type I and define

$$G_x(y) = \begin{cases} 1 & \text{if } x < y \\ 0 & \text{if } x \geq y \end{cases} \tag{3.2}$$

Suppose now that $y \in \Gamma$ is of type II. Proposition 3.1 ensures the existence of an interval I with end-points a and b such that $\Lambda_{q_1,q_2} = \{W(y) = k(y)\}$ a.s. for any $y \in I$. We relabel Λ_{q_1,q_2} as $\Lambda(a,b)$ and define

$$G_x(y) = \begin{cases} 1 & \text{if } x < a \\ \dfrac{F(y)-q_1}{P(\Lambda(a,b))} & \text{if } a \leq x < b \\ 0 & \text{if } x \geq b \end{cases} \tag{3.3}$$

When a and/or b are infinite, (3.3) undergoes obvious modifications. Let $U = \{x: x = \lim_{k\to\infty} x_{n_k}$ for some $x_{n_k} \in \operatorname{supp} F_{n_k}\}$. In U we have included subsequences $\{x_{n_k}\}$ to make allowance for the case when for some $x \notin \operatorname{supp} F$

for sequences converging in distribution.

Proposition 3.1. Suppose that y is of type II. Then the interval I is the maximal set of points z of type II containing y such that $\{W(y) = k(y)\} = \{W(z) = k(z)\}$ a.s. The point a belongs to I if and only if a is of type II_1; a and b are of type I'; and (a,b) contains only points of type II_2.

Proof. We show first that (a,b) does not contain points of type I'. Assume the contrary, that $\hat{y} \in (a,b)$ is of type I'. Then $q < F(\hat{y}-) \leq P(W(\hat{y})=1) \leq F(\hat{y}) < q_2$. However, by Lemma 2.3(a) $W_{\hat{q}}$ is of type I with $\hat{q} = P(W(\hat{y}) = 1)$ which is in contradiction with Lemma 2.3 (b).

We prove next that a is of type I'. Let $P(W(y) = 1) = F(a)$. Since $W(y)$ may take at most three distinct values with positive probability, one can choose z such that by Proposition 2.1(a) $\lim_{n\to\infty} \{X_n \in I_n\} = \{W(y) > z\} = \{W(y) = 1\}$ a.s.

and Lemma 2.2(b) implies that a is of type I. Notice that in this case $a \notin I$ by the way a was defined, which agrees with the statement that I contains only points of type II. If $q_1 < F(a)$ then $F(a-) \leq q_1 = P(W(y) = 1) < F(a)$. Notice now that $W(z)$ is a.s. right-continuous because $W(z)$ is monotone in z and $E(W(z + \varepsilon) - W(z)) = F(z + \varepsilon) - F(z) \to 0$ as $\varepsilon \to 0$ is due to the right-continuity of F. Thus either $y = a$ or $y > a$ and taking the limit of $P(W(y) = 1)$ as $y \to a$ we get from the above inequality that $F(a-) \leq P(W(a) = 1) < F(a)$, proving that a is of type II_1.

We prove next that b is of type I'. If $q_2 = P(W(y) \geq k(y)) = F(b)$ we can argue as in the case of a to show that b is of type I. If $q_2 < F(b)$ then b may be of type I or II. If b is of type I there is nothing left to prove. If

We note that the condition assumed in Theorem 2.4 on F' and F'' is always satisfied if F' is continuous.

Corollary 2.2. Suppose that $\{X_n\}$ converges weakly and contains a subsequence converging in probability. Then $\{X_n\}$ converges in probability.

3. Sequences converging in distribution. We shall now assume that $\{X_n\}$ converges in distribution to a non-degenerate limit F. Define

$$F_x^{(n)}(y) = \lim_{m\to\infty} P(X_m \le y | X_n = x) \tag{3.1}$$

where y is a continuity point of F and n = 0,1,... . Theorem 2.1(a) ensures the existence of $\{F_x^{(n)}(y)\}$, which may be extended to right-continuous functions with respect to y by defining $F_x^{(n)}(y) = \lim_{y'\downarrow y} F_x^{(n)}(y')$ for any jump point y of F.

We agree to write W(y) for W_q with q = F(y) and define y to be of type I or II according as W(y) is of type I or II. If F admits jump points, there must be values of q for which there is no y with F(y) = q even if W_q may be well defined, since $q = \lim_{n\to\infty} P(X_{n_k} \in J_{n_k})$ for some sequence $\{n_k\}$ and left-unbounded intervals $\{J_{n_k}\}$ is a possibility. A point y of type II will be said to be of type II_1 if $F(y-) \le P(W(y) = 1) < F(y)$, and of type II_2 otherwise. Points of type I or II_1 will be said to be of type I'.

Assume that y is of type II and the possible values of W(y) are 0,k(y) and 1. Write as before $q_1 = P(W(y) = 1)$ and $q_2 = P(W(y) = 1) + P(W(y) = k(y))$ and define $a = \inf\{x: x \in \text{supp } F, F(x) > q_1\}$ and $b = \inf\{x: x \in \text{supp } F, F(x) \ge q_2\}$. Let I be (a,b) or [a,b) according as $F(a) = q_1$ or $> q_1$. Of course, a and/or b may be infinite. The next result characterizes points of type II

$\{X_{n_k'}\}$ and $\{X_{n_k''}\}$ under the assumption that $\{X_{n_k'}\}$ converges in probability. For a weakly convergent sequence $\{X_n\}$ we define convergence in probability to a not necessarily finite X as the fulfilment of the condition:

$\lim_{n\to\infty} P(\{X_n \leq x\} \wedge \{X \leq x\}) = 0$ for any continuity point x of F where F is the distribution function of X.

Theorem 2.4. Suppose that $\{X_n\}$ is a SM Markov chain and there exists a subsequence of $\{X_n\}$, say $\{X_{n_k'}\}$, converging in probability to a not necessarily a.s. finite random variable X, and that $\{X_{n_k''}\}$ is another subsequence of $\{X_n\}$, converging weakly. If $\{y: F'(x) = y,\ x \in C(F')\} \supseteq \{y: F''(x) = y, x\in C(F'')\}$ where F' and F" and the limit distributions of $\{X_{n_k'}\}$ and $\{X_{n_k''}\}$, and C(F') and C(F'') are the sets of their continuity points, then $\{X_{n_k''}\}$ converges in probability.

Proof. Choose x to be a continuity point of F'' and write $F''(x) = q$. Then there must be a continuity point of F', say x', such that $F'(x') = q$. Since $\{X_{n_k'}\}$ was supposed to be convergent in probability, it contains an a.s. convergent sequence, so that we can assume the existence of $\lim_{k\to\infty} \{X_{n_k'} \leq x'\}$ a.s. In view of Lemma 2.2(b) this makes W_q of type I. Further, by Lemma 2.2(a) there exists a sequence of left-unbounded intervals $\{I_n\}$ such that $\lim_{n\to\infty} \{X_n \in I_n\} = \{W_q = 1\}$ a.s. with $\lim_{n\to\infty} P(X_n \in I_n) = q$. It follows that $\lim_{k\to\infty} P(\{X_{n_k''} \in I_{n_k''}\} \wedge \{X_{n_k''} \leq x\}) = 0$ and by transitivity $\lim_{k\to\infty} P(\{X_{n_k''} \leq x\} \wedge \{W_q = 1\}) = 0$. An appeal to Lemma 2 of [7] yields now convergence in probability for $\{X_{n_k''}\}$.

where x is a continuity point of F, B is an event in the σ-field generated by $X_0,\ldots,X_m$ and m an arbitrary non-negative integer. We shall consider a generalization of (2.4) to Markov chains which are not necessarily convergent in distribution: $\{X_n\}$ will be said to be mixing if for any $q \in (0,1)$ for which there exist a subsequence $\{n_k\}$ and left-unbounded intervals $\{J_{n_k}\}$ such that $\lim_{k\to\infty} P(X_{n_k} \in J_{n_k}) = q$, then

$$(2.5) \qquad \lim_{k\to\infty} P(X_{n_k} \in J_{n_k} | X_n) = q \text{ a.s. for } n = 0,1,\ldots$$

Theorem 2.3. Suppose that $\{X_n\}$ is a SM Markov chain and there exist $\hat{q} \in (0,1)$, some numbers $\{\hat{n}_k\}$ and left unbounded intervals $\{J_{\hat{n}_k}\}$ such that $\lim_{k\to\infty} P(X_{\hat{n}_k} \in J_{\hat{n}_k} | X_{\hat{n}_m}) = \hat{q}$ a.s. for $m = 0,1,\ldots$. Then $\{X_n\}$ is mixing.

Proof. The condition stated is equivalent to $W_{\hat{q}} = \hat{q}$ a.s. for some $\hat{q}$ in (0,1). This implies $\hat{q}_1 = P(W_{\hat{q}} = 1) = 0$ and $\hat{q}_2 = 1 - P(W_{\hat{q}} = 0) = 1$. By Lemma 2.3(b) there are no points of type I in (0,1), i.e. there are no points of type I at all, which leads to $P(W_q = 0) = P(W_q = 1) = 0$ for any q. It follows that $W_q = q$ a.s. and an appeal to Theorem 2.1 yields (2.5) completing the proof.

Theorem 2.3 expresses the rather surprising property that mixing is ensured merely if (2.5) holds for one value q in (0,1) and some $\{n_k\}$ and $\{J_{n_k}\}$.

The next result relates properties of two weakly convergent sequences

Proof. By Lemma 2.2(a) for any $\varepsilon > 0$ there exist some left-unbounded intervals $\{J_n(q-\varepsilon)\}$, $\{J_n(q)\}$ and $\{J_n(q+\varepsilon)\}$ such that $\lim_{n\to\infty} \{X_n \in J_n(q-\varepsilon)\} = \{W_{q-\varepsilon} = 1\}$ a.s., $\lim_{n\to\infty} \{X_n \in J_n\} = \{W_q = 1\}$ a.s. and $\lim_{n\to\infty} \{X_n \in J_n(q+\varepsilon)\} = \{W_{q+\varepsilon} = 1\}$ a.s. Since for n large enough $J_n(q-\varepsilon) \subset (-\infty, x_n] \subset J_n(q+\varepsilon)$, it follows that $\{W_{q-\varepsilon} = 1\} \subset \liminf_{n\to\infty} \{X_n \le x_n\} \subseteq \limsup_{n\to\infty} \{X_n \le x_n\} \subset \{W_{q+\varepsilon} = 1\}$. But $\lim_{\varepsilon\to 0} P(W_{q-\varepsilon} = 1) = \lim_{\varepsilon\to 0} P(W_{q+\varepsilon} = 1) = P(W_q)$, concluding the proof.

Theorem 2.2. Suppose that $\{X_n\}$ is a SM Markov chain and that $\{W_q\}$ exist for all $q \in (0,1)$ and are of type I. If $\{f_n\}$ are some non-decreasing measurable functions such that $\{Y_n\}$, with $Y_n = f_n(X_n)$, converges weakly, then $\{Y_n\}$ converges a.s.

Proof. To prove a.s. convergence for $\{Y_n\}$ it suffices to show that $\lim_{n\to\infty} \{Y_n \le x\}$ a.s. exists for any continuity point x of the limit distribution $\hat{F}$ of $\{Y_n\}$, including points x for which $\hat{F}(x) = 1$ or $\hat{F}(x) = 0$, (which may be the only points of this kind when $\hat{F}$ is degenerate). Since there are always points $\{x_n\}$ such that $\{Y_n \le x\} = \{X_n \le x_n\}$, an appeal to Lemma 2.5 finishes the proof.

The next result refers to $\{W_q\}$ of type II for which $P(W_q = k_q) = 1$. Such a case is related to a condition of mixing given by Renyi [23]. A sequence $\{X_n\}$ converging in distribution to a limit F is said to be mixing in the sense of Renyi if

$$\lim_{n\to\infty} P(\{X_n \le x\} \cap B) = F(x)P(B) \tag{2.4}$$

$P(W_q = k_q) > 0$. By Lemma 2.3(a) $q_1 = P(W_q = 1)$ and $q_2 = 1-P(W_q = 0)$. It follows that $E(W_q) = q = q_1 + k_q P(W_q = k_q) = q_1 + k_q(q_2-q_1)$. This yields $k_q = (q-q_1)/(q_2 - q_1)$ and $W_q = 1_{\{W_{q_1}=1\}} + (q - q_1)/(q_2 - q_1)\, 1_{\Lambda_{q_1,q_2}}$ a.s. is now easily derived from the remaining statement of Lemma 2.3(a). To complete the proof notice that since W_{q_1} and W_{q_2} are of type I, Lemma 2.3(b) makes it impossible for (q_1,q_2) to vary with q.

Remark 2.1. The statement about the existence of $G_{X_n}^{(n)}(q)$ in Theorem 2.1 (a) contains as particular cases Proposition (3.1)(j) and Remark (3.2) of [2] removing the restriction (1.1).

Remark 2.2. Although we defined W_q in relation to a subsequence $\{n_k\}$ for which there exist intervals $\{J_{n_k}\}$ such that $q = \lim_{k\to\infty} P(X_{n_k} \in J_{n_k})$, the variables $\{W_q\}$ turned out to be independent of subsequence choice. If W_q is of type I we have seen that there must exist a whole sequence $\{I_n\}$ of left-unbounded intervals such that $\lim_{n\to\infty} P(X_n \in I_n) = q$ whereas for W_q of type II this need not happen. However, if condition (1.1) is imposed we can define $c_q^{(n)}$ to be the q-quantiles of X_n and get that $\lim_{n\to\infty} P(X_n \le c_q^{(n)}) = q$ for any $q \in (0,1)$ so that in this case W_q exists for all q and, besides, there is no need to confine ourselves to limits of subsequences when defining $\{G_X^{(n)}(q)\}$.

Lemma 2.5. Suppose that $\{W_q\}$ exist for all $q \in (0,1)$ and are of type I. Then $\lim_{n\to\infty} \{X_n \le x_n\} = \{W_q=1\}$ a.s. for any $\{x_n\}$ such that $\lim_{n\to\infty} F_n(x_n) = x$.

such that $\lim_{k\to\infty} P(X_{n_k} \in J_{n_k}) = q$ for some sequence $\{n_k\}$ and left-unbounded intervals $\{J_{n_k}\}$. Then

(a) There exist the random variables $G^{(n)}_{X_n}(q) = \lim_{k\to\infty} P(X_{n_k} \in J_{n_k} | X_n)$ a.s. and $W_q = \lim_{n\to\infty} G^{(n)}_{X_n}(q)$ a.s. for $n = 0,1,\ldots$ where $E(W_q|X_n) = G^{(n)}_{X_n}(q)$ a.s. for $n = 0,1,\ldots$ and $E(W_q) = q$.

(b) The variables W_q are of two possible types: I, if $W_q = 1_{\{W_q = 1\}}$ a.s., and II, if $W_q = 1_{\{\hat{W}_{q_1} = 1\}} + (q-q_1)/(q_2-q_1)\, 1_{\Lambda_{q_1,q_2}}$ a.s., where $\Lambda_{q_1,q_2} = \{\hat{W}_{q_2}=1\} \setminus \{\hat{W}_{q_1} = 1\}$, $0 \le q_1 < q < q_2 \le 1$, $\hat{W}_{q_1} = 0$ or W_{q_1} (of type I) according as $q_1 = 0$ or > 0, and $\hat{W}_{q_2} = 0$ or W_{q_2} (of type I) according as $q_2=1$ or < 1. The quantities q_1 and q_2 do not depend on the choice of q in (q_1,q_2).

Proof. According to Lemma 2.1 $G^{(n)}_{X_n}(q) = E(W_q|X_n)$ a.s., which in view of Lemma 2.4 does not depend on the choice of $\{n_k\}$ and $\{J_{n_k}\}$ such that $\lim_{k\to\infty} P(X_{n_k} \in J_{n_k}) = q$. Thus any subsequence of $\{P(X_{n_k} \in J_{n_k}|X_n),\ k = 1,2,\ldots\}$ contains a further subsequence converging to $G^{(n)}_{X_n}(q)$. It follows that the whole subsequence $\{P(X_{n_k} \in J_{n_k}|X_n),\ k = 1,2,\ldots\}$ converges to the same limit $G^{(n)}_{X_n}(q)$. The remaining statement in (a) follows from Lemma 2.1.

To prove (b) recall first that according to Proposition 2.1(b) W_q may take at most three distinct values with positive probability. If $P(W_q = 1) = 1 - P(W_q = 0)$ W_q is said to be of type I, in which case it is obvious that $W_q = 1_{\{W_q=1\}}$ a.s. with $E(W_q) = P(W_q = 1) = q$, whereas in the case when W_q is of type II there are three possible values for W_q: $0, k_q$ and 1 with an i-atom if either $P(\Lambda) = 0$ or $P(\Lambda) = P(A)$ for any Λ such that

$\Lambda = \lim_{n\to\infty} \{X_n \in L_n\}$ a.s. for some intervals $\{L_n\}$ with $L_n \subset J_n$, $n=0,1,\ldots$

Lemma 2.3 (b) yields the following.

Corollary 2.1 If W_q is of type II then $\{W_q = k_q\}$ is an i-atom.

The sets $\{W_q = k_q\}$ will turn out to correspond to the sets $\{\Lambda_i\}$ described in Fig. 1.1 for the sequences $\{X_n\}$ convergent in distribution.

Lemma 2.4 The random variables $\{W_q\}$ do not depend on the choice of $\{n_k\}$ and $\{J_{n_k}\}$ such that $\lim_{k\to\infty} P(X_{n_k} \in J_{n_k}) = q$.

Proof. Choose two subsequences $\{n_k\}$ and $\{n_k'\}$ such that

$\lim_{k\to\infty} P(X_{n_k} \in J_{n_k}) = \lim_{k\to\infty} P(X_{n_k'} \in J_{n_k'}) = q$ for some left-unbounded intervals $\{J_{n_k}\}$ and $\{J_{n_k'}\}$, and construct the limit variables W_q and W_q' corresponding to the two subsequences. Assume first that W_q is of type I. Then Lemma 2.2(a) and (b) may be invoked to show that W_q' is also of type I, and a reasoning may be easily extracted from the proof of Lemma 2.2(b) to yield that $\{W_q = 1\} = \{W_q' = 1\}$ a.s. This is equivalent to $W_q = W_q'$ a.s. which finishes the proof in case I.

Assume now that W_q is of type II. Then W_q' is also of type II. Recall the notation used in Lemma 2.3 and write q_1 and q_2 for the quantities attached to W_q and q_1' and q_2' for the quantities attached to W_q'. Since $q_1 < q < q_2$ and $q_1' < q < q_2'$, (q_1,q_2) and (q_1',q_2') may either partly overlap or coincide. Since W_{q_1}, W_{q_2}, $W_{q_1'}'$, $W_{q_2'}'$ are of type I, we get $W_{q_1'}' = W_{q_1'}$ and $W_{q_2'}' = W_{q_2'}$ by the proven part of Lemma 2.4 and an inspection of Lemma 2.3 is easily seen to lead to $q_1 = q_1'$ and $q_2 = q_2'$, completing the proof.

Theorem 2.1. Suppose that $\{X_n\}$ is a SM Markov chain and choose $q \in (0,1)$

useful in deriving some properties of F as strict monotonicity, continuity, finiteness of moment, etc. (see [8]).

Remark 3.3. If all $y \in \Gamma$ are of type I, $\{X_n\}$ may not converge a.s. By Theorem 2.2 convergence a.s. may fail if not all $\{W_q\}$ are of type I. Such a situation may arise if F admits jump points resulting from lumping together some values of q for which W_q are of type II. Also if F has intervals on which it is constant, Theorem 3.2(a) cannot be invoked to get $\lim_{n\to\infty} \{X_n \in I_n\} = \lim_{n\to\infty} \{X_n \leq y\}$ a.s. for any $y \in \Gamma$ as done in [7] when proving a.s. convergence. The minimal condition on F guaranteeing a.s. convergence seems to be:

(b) F is either continuous or admits jump points $\{c_i, i \in \Theta\}$ such that $(F(c_i + \delta) - F(c_i))(F(c_i-) - F(c_i - \delta)) > 0$ for any $\delta > 0$ and $i \in \Theta$. This condition was considered in [7] and shown to entail the equivalence of a.s. convergence and convergence in probability.

4. The stationary transition probability case. Assume that $\{Y_n\}$ is a chain with stationary transition probabilities, $\{a_n\}$ with $a_n > 0$ and $\{b_n\}$ are two sequences of constants making $\{a_n(Y_n + b_n)\}$ convergent in distribution to a non-degenerate limit F. Write $X_n = a_n(Y_n + b_n)$ for $n = 0,1,\ldots$, $F_x(y) = \lim_{n\to\infty} P(X_n \leq x | Y_0 = x)$ where y is a continuity point of F, and $\nu_n(\cdot) = P(Y_n \in \cdot)$ for $n = 0,1,\ldots$ Further $\nu << \mu$ is to denote that ν is absolutely continuous with respect to μ.

Lemma 4.1. Suppose that $\nu_1 << \nu_0$. Then there exist two constants α and β such that $\lim_{n\to\infty} a_{n+1}/a_n = \alpha$ and $\lim a_{n+1}(b_{n+1}-b_n) = \beta$ with $0 < \alpha < \infty$ and $-\infty < \beta < \infty$.

Proof. Let y be a continuity point of F and

$$(4.1) \qquad P(X_n \leq y) = \int P(X_n \leq y | Y_0 = x)\nu_0(dx)$$

The dominated convergence theorem applied to (4.1) yields

$$(4.2) \qquad F(y) = \lim_{n\to\infty} P(X_n \leq y) = \int F_x(y)\nu_0(dx)$$

Using the stationarity of transition probabilities one gets

$$(4.3) \qquad P(a_n(Y_{n+1} + b_n) \leq y) = \int P(Y_{n+1} \leq y/a_n - b_n | Y_1 = x)\nu_1(dx)$$

$$= \int P(X_n \leq y | Y_0 = x)\nu_1(dx)$$

$$= \int P(X_n \leq y)|Y_0 = x)\lambda\nu_0(dx)$$

where $\lambda = d\nu_1/d\nu_0$ stands for the Radon-Nycodym derivative of ν_1 with respect to ν_0. The dominated convergence theorem applied to (4.3) yields the existence of $F_1(y) = \lim_{n\to\infty} P(a_n(Y_{n+1} + b_n) \leq y)$ and

$$(4.4) \qquad F_1(y) = \int F_x(y)\lambda\nu_0(dx)$$

Since ν_0 $(0 < \lambda < \infty) = 1$ one may easily see that F_1 is not degenerate. Indeed, if $F_1(y)$ where 0 or 1 according as $y < c$ or $y \geq c$ for a certain constant c, then by (4.4) the same property would hold for $F_x(y)$ for almost all x with respect to ν_0 and by (4.2) F would be degenerate as well. Thus both $\{a_n(Y_n + b_n)\}$ and $\{a_n(Y_{n+1} + b_n)\}$ converge in distribution to non-degenerate limits and the result now follows from Khintchine's theorem on convergence of types (see e.g. [22] p. 2̇6).

Lemma 4.1 is an improvement on Theorem 3 of [5] where a stronger condition was assumed on F. We notice that stochastic monotonicity was not used in the proof.

Lemma 4.2. Suppose that the conditions of Lemma 4.1 are satisfied with $\alpha \neq 1$. Then there exist some constants $\{b_n'\}$ such that $\{a_n(Y_n + b_n')\}$ converges in distribution to a non-degenerate limit and $\lim_{n\to\infty} a_{n+1}(b_{n+1}' - b_n') = 0$.

Proof. Take $b_n' = b_n - \lambda_0/a_n$ where $\lambda_0 = \beta/(1 - \alpha)$. Since $a_n(Y_n + b_n') = X_n - \lambda_0$, convergence in distribution of $\{a_n(Y_n + b_n')\}$ to a non-degenerate limit clearly obtains. Further $a_{n+1}(b_{n+1}' - b_n') = a_{n+1}(b_{n+1} - b_n) - \lambda_0 + a_{n+1}/a_n\lambda_0$, and taking limits gives $\lim_{n\to\infty} a_{n+1}(b_{n+1}' - b_n') = \beta - \lambda_0 + \alpha\lambda_0 = 0$, finishing the proof.

Remark 4.1. Suppose that $\beta = 0$ and x_0 is a continuity point of F. Then $\alpha^n x_0$ is also a continuity point of F for any integer n. Indeed, using Lemma 4.1 in (4.4) yields $F_1(x) = F(\alpha\, x)$ and

$$(4.5) \qquad F(\alpha(x_0 + \varepsilon)) - F(\alpha(x_0 - \varepsilon)) = \int (F_x(x_0 + \varepsilon) - F_x(x_0 - \varepsilon))\lambda\nu_0(dx)$$

On the other hand (4.2) implies

$$(4.6) \qquad F(x_0 + \varepsilon) - F(x_0 - \varepsilon) = \int (F_x(x_0 + \varepsilon) - F_x(x_0 - \varepsilon))\nu_0(dx)$$

and it is clear that if F is continuous at x_0 and we let $\varepsilon \to 0$ in (4.6), the integrand must tend to 0 as well for almost all x with respect to ν_0, i.e. $F_x(\cdot)$ turns out to be continuous at x_0 for almost all x with respect to ν_0. By (4.5) this implies that x_0 is a continuity point of F if and only if αx_0

is a continuity point of F and therefore if x_0 is a continuity point of F, so will be $\alpha^n x_0$ for any integer n.

Similarly, one may show that $F(x_2) - F(x_1) = 0$ for $x_2 > x_1$ entails $F(\alpha^n x_2) - F(\alpha^n x_1) = 0$ for any integer n.

Let Θ be the shift function defined on Ω by $\Theta(\omega_0, \omega_1, \ldots) = (\omega_1, \omega_2, \ldots)$ and write $\Theta\Lambda = \{\Theta\omega : \omega \in \Lambda\}$, $\Theta^0\Lambda = \Lambda$, $\Theta^{-1}\Lambda = \{\omega : \Theta\omega \in \Lambda\}$, $\Theta^k\Lambda = \Theta(\Theta^{k-1}\Lambda)$ and $\Theta^{-k}\Lambda = \Theta^{-1}(\Theta^{-k+1}\Lambda)$ for $k = 1,2,\ldots$. If J is an interval with end-points x_1 and x_2, $\hat{\Theta}J$ is to denote the interval obtained from J by replacing x_1 and x_2 by αx_1 and αx_2 respectively.

We shall further need the following.

<u>Lemma 4.3</u>. Suppose that $\nu_1 << \nu_0$ and that for some left-unbounded intervals $\{J_n\}$ $\lim_{n\to\infty} \{Y_n \in J_n\}$ a.s. exists. Then $\lim_{n\to\infty} \{Y_{n+k} \in J_n\}$ a.s. also exists and $\Theta^k \lim_{n\to\infty} \{Y_n \in J_n\} = \lim_{n\to\infty} \{Y_{n+k} \in J_n\}$ a.s.

This result can be extracted from Theorem 5 of [1] and Lemma 2 p. 91 of [6].

<u>Theorem 4.1</u>. Suppose that $\{Y_n\}$ is a SM Markov chain with stationary transition probabilities, $\nu_1 << \nu_0$ and $\{X_n\}$ with $X_n = a_n(Y_n + b_n)$ converges in distribution to a non-degenerate limit F with $\alpha \neq 1$. Then, if necessary after a recentering, $\beta = 0$ and

(a) there exists at least one point y_0 of type I'

(b) if $y_0 \neq 0$ is of type $I(II_1)$ then $\alpha^n y_0$ is also of type $I(II_1)$ for all n

(c) if $y_0 \neq 0$ then any interval J of points of type II_2 of the same sign as y_0 is contained in an interval $(\alpha^n y_0, \alpha^{n+1} y_0)$ for $y_0 > 0$ (or $(\alpha^{n+1} y_0, \alpha^n y_0)$ for $y_0 < 0$) for some integer n. If J is an interval of point of type II_2 then $\hat{\Theta}^n J$ is also an interval of points of type II_2 for all n.

Proof. Notice that by Lemma 4.2 one may take $\beta = 0$. Choose y to be a continuity point of F. Then by Remark 4.1 αy is also a continuity point of F and

$$(4.7) \qquad F_{x_0}^{(0)} = \lim_{m\to\infty} P(X_m \le y \mid Y_0 = x)$$

$$= \lim_{m\to\infty} P(a_m(Y_{m+1} + b_m) \le y \mid Y_1 = x)$$

$$= F_{x_1}^{(1)}(\alpha y)$$

where $x_0 = a_0(x + b_0)$ and $x_1 = a_1(x + b_1)$. Assume by way of contradiction that there are no points of type I', case that occurs only if $W(y)$ is a.s. constant for all y. Since $E(W(y)|X_n) = F_{X_n}^{(n)}(y)$ a.s. it follows that $F_{X_n}^{(n)} = E(W(y)) = F(y)$ a.s. Using this in (4.7) for $n = 0$ and 1 gives $F(y) = F(\alpha y)$. This leads to $F(y) = F(\alpha^n y)$ for all n and if $y > 0$ one gets $F(y) = 1$ whereas if $y < 0$ one gets $F(y) = 0$. But such F is degenerate and we reached a contradiction that proves that there exists at least one point y_0 of type I' and (a) is proved.

We prove (b) for y_0 of type I (for type II_1 the proof is similar). By Theorem 3.1(a) there exist some left-unbounded intervals $\{I_n\}$ such that $T = \lim_{n\to\infty} \{X_n \in I_n\}$ a.s. and $P(T) = F(y_0)$. Further by Lemma 4.3 $\Theta^n T = \lim_{m\to\infty} \{\hat{X}_{m+n} \in I_m\}$ a.s. exists for all n, where $\hat{X}_{m+n} = a_n(Y_{m+n} + b_n)$. It remains to prove that $\lim_{m\to\infty} P(\hat{X}_{m+n} \in I_m) = F(\alpha^n y_0)$ which we shall confine ourselves to prove for $n = 1$. By (4.3) and (4.4) we get

(4.8) $|F(\alpha x) - P(\hat{X}_{m+1} \epsilon I_m)| \leq \int |F_x(y) - P(X_m \epsilon I_m | Y_0 = x)| \lambda\nu_0 \, (dx)$

But $F_{X_n}^{(n)}(y) = \lim_{m\to\infty} P(X_m \epsilon I_m | X_n)$ a.s. for all n, and taking n = 0 we get

$\lim_{m\to\infty} |F_x(y) - P(X_m \epsilon I_m | Y_0 = x)| = 0$ for almost all x with respect to ν_0.

Using this in (4.8) completes the proof of (b).

To prove (c) notice that there is no interval of points of type II_2 straddling $(\alpha^n y_0, \alpha^{n+1} y_0)$ for some n. Indeed, this is included by the fact that $\alpha^n y_0$ and $\alpha^{n+1} y_0$ are of type I'. Further, if J is an interval of points of type II_2, then $\hat{\Theta}^n J$ must also be an interval of points of type II_2, since otherwise if y were of type I' with $y \epsilon \hat{\Theta}^n J$ then $\alpha^{-n} y \epsilon J$ and by (b) $\alpha^{-n} y$ would be of type I'. This contradiction completes the proof.

Corollary 4.1

(a) If $(-\infty, 0)$ does not contain any point of type I', then $F(0-) = 0$

(b) If $(0, \infty)$ does not contain any point of type I', then $F(0) = 1$.

Proof. (a) and (b) being symmetric, it will suffice to prove (a). We show first that $J = (-\infty, 0)$ is the maximal interval of points of type II_2 containing y with $y < 0$. Indeed, according to Theorem 4.1 either 0 is of type I' or there are points of type I' in $(0, \varepsilon)$ for any $\varepsilon > 0$, in which case Proposition 3.1 implies that 0 is of type I' and $J = (-\infty, 0)$. It follows that $q_1 = 0$ and $F(y) = E(W(y)) = k(y) q_2$ for $y < 0$. Consider now the sequence $Z_n = a_n(Y_{n+1} + b_n)$ and agree to attach the prime to the symbols for $\{X_n\}$ when referring to $\{Z_n\}$. We claim that $J' = (-\infty, 0)$. Indeed, since the limit distribution of $\{Z_n\}$ is $F'(x) = F(\alpha x)$, Theorem 4.1(b) implies that $\{X_n\}$ and $\{Z_n\}$ must assume the same points of type I, II_1 and II_2. Since $J = (-\infty, 0)$ we get $J' = (-\infty, 0)$ as well. We show now that $k'(y) = k(y)$ for $y < 0$. Indeed, by the stationarity of the transition probabilities of $\{Y_n\}$ we get $P(Z_n \leq y \mid Z_m = x) = P(X_n \leq y \mid X_m = x)$ for $n > m$ and recalling the definitions of $\{F_x^{(n)}(y)\}$, $\{F_x'^{(n)}(y)\}$ and Theorem 3.1 we get $k(y) = k'(y)$ for $y < 0$. Recall that either of $\{W_{q_2} > 0\}$ or $\{W'_{q_2'} > 0\}$ may be expressed as $\lim_{n\to\infty} \{Y_n \in J_n\}$ a.s. for some intervals $\{J_n\}$, and by Lemma 2.3(a) both W_{q_2} and W'_{q_2} are of type I. These considerations in conjunction with Lemmas 2.2(b) and 2.3(b) boil down to $q_2' = q_2$. It follows that $F'(y) = E(W'(y)) = k(y)q_2 = E(W(y)) = F(y)$ for $y < 0$ which is incompatible with $F'(x) = F(\alpha x)$ for $\alpha \neq 1$ unless $F(0-) = 0$ and the proof is finished.

Remark 4.2. An interesting consequence of Corollary 4.1 is that $y_0 \neq 0$ of type I' always exists. This property in conjunction with Theorem 4.1 leads to the conclusion that 0 is also a point of type I'. Another consequence of Theorem 4.1 is that any interval $(-\varepsilon,\varepsilon)$ with $\varepsilon > 0$ contains all the information concerning the points of type I, II_1 and II_2 of the real line. In particular, if $\lim_{n\to\infty} \{X_n \leq x\}$ a.s. exists for $x \in (-\varepsilon,\varepsilon)$ then $\{X_n\}$ converges a.s.

Remark 4.3. The case $\alpha=1$, $\beta\neq 0$ may be treated in a similar way, taking into account that $\alpha^n \lim_{m\to\infty} \{X_m \in I_m\} = \lim_{m\to\infty} \{X_m \in I_{m,n}\}$ a.s. where $I_{m,n}$ is obtained from I_m by replacing y_m with $y_m + n\beta$ (see [6]). Theorem 4.1(a) carries over without changes. For Theorem 4.1(b) and (c), the requirement $y_0 \neq 0$ is no longer necessary whereas $\alpha^n y_0$, $\alpha^n a$ and $\alpha^n b$ are replaced by $y_0 + n\beta$, $a + n\beta$ and $b + n\beta$ respectively. Corollary 4.1 may also be extended to this case on using a similar proof.

Theorem 4.2. Suppose that $\{Y(t): t \in [0,\infty)\}$ is a right-continuous Markov process with stationary transition probabilities, $a(t)$ and $b(t)$ some continuous, monotone functions with $\lim_{t\to\infty} a(t) = 0$ or ∞ such that $X(t)=a(t)(Y(t)+b(t))$ converges in distribution to a non-degenerate limit F. Assume that $\nu_t << \nu_s$, where $\nu_t(\cdot) = P(Y(t) \in \cdot)$. Then $\lim_{t\to\infty} a(t+s)/a(t)=\rho^s$ and $\lim_{t\to\infty} a(t+s)(b(t+s)-b(t))=\beta s$ for some constants ρ and β and all $s > 0$. In addition, one of the following cases occurs:

(a) $\rho=1$ and $\beta=0$. If in addition, $\lim_{t\to\infty} P(X(t) \leq x|Y(0)=y) = F(x)$ for all x and y, then $P(\liminf X(t) \leq s_1) = P(\limsup X(t) \geq s_1) = 1$, where $s_1=\inf \operatorname{supp} F$ and $s_2 = \sup \operatorname{supp} F$.

(b) either $\rho\neq 1$ or $\beta \neq 0$, in which case there exists a random variable W such that $\lim_{t\to\infty} X(t) = W$ a.s. In addition, supp F is either the real line of one of its half-lines, and F is strictly increasing on its support. If $\rho = 1$, F is continuous, whereas if $\rho \neq 1$ and $\beta = 0$, F is continuous except may be for $x = 0$.

Proof. We shall first show that $\lim_{t\to\infty} a(t + s)/a(t) = \rho^s$ for some constant ρ and all $s > 0$. To this aim let us consider the skeleton chain $\{X(n\delta):n\geq 0\}$ for a certain $\delta > 0$. According to Lemma 4.1 $\lim_{n\to\infty} a((n+1)\delta)/a(n\delta) = \alpha(\delta)$ exists. Take further $\delta' = \delta/k$ for a positive integer k and write $\alpha(\delta') = \lim_{n\to\infty} a((n + 1)\delta')/a(n\delta')$. But $a((n + 1)\delta)/a(n\delta) =$

$a((n + 1)\delta)/a((n+ 1) -\delta')a((n - 1) -\delta')/a((n + 1)\delta-2\delta')\ldots a(n\delta+\delta')/a(n\delta)$ and taking $n\to\infty$ we get $\alpha(\delta) = \alpha^k(\delta')$. Also, it is easy to see that if k' is a positive integer then $\alpha(k'\delta') = \alpha^{k'}(\delta')$ and therefore $\alpha(k'\delta/k) = \alpha^{k'/k}(\delta)$. Thus for any rational number $r > 0$, $\alpha^r(\delta) = \alpha(r\delta)$. Consider further an arbitrary number s and write $h(n\delta) = a(n\delta + s)/a(n\delta)$. We shall show that $\lim_{n\to\infty} h(n\delta)$ exists for all $\delta > 0$. Indeed, $\lim_{n\to\infty} h(n\delta)$ exists for $\delta = s$ as shown above. Consider now $s' = s/k$ for a positive integer k and $h'(n\delta) = a(n\delta + s')/a(n\delta)$. As above we can show that $\lim_{n\to\infty} h(ns) = (\lim_{n\to\infty} h'(ns'))^k$. Thus $\lim_{n\to\infty} h(ns) = \lim_{n\to\infty} a(ns' + s)/a(ns') = \lim_{n\to\infty} h(ns')$. It is easy to see that we can replace here s' by a multiple of s' and therefore $\lim_{n\to\infty} h(n\delta)$ exists and does not depend on δ for $\delta = rs$ where r is any rational and positive number. Choose now $\delta_1 = r_1 s$ and $\delta_2 = r_2 s$ with r_1 and r_2 rational such that $0 < \delta_1 < \delta < \delta_2 < \infty$. By the monotonicity of a(t) $h(n\delta_1) \leq h(n\delta) \leq h(n\delta_2)$ and since $\lim_{n\to\infty} h(n\delta_1) = \lim_{n\to\infty} h(n\delta_2)$ one gets that $\lim_{n\to\infty} h(n\delta)$ exists for all $\delta > 0$. We are now in a position to invoke a result by Kingman [18] asserting that if $\lim_{n\to\infty} h(n\delta)$ exists for all $\delta > 0$ and h is continuous then $\lim_{t\to\infty} h(t)$ also exists. We have already proved that $\alpha^s(\delta)=\alpha(s\delta)$ for s rational. It is easy to see that this equality extends to any $s > 0$, and taking $\alpha(1)=\rho$ we get $\lim_{t\to\infty} h(t) = \rho^s$. A similar reasoning yields $\lim_{t\to\infty} a(t+s)(b(t+s)-b(t)) = \beta s$ where $\beta = \lim_{t\to\infty} a(t + 1)(b(t + 1) - b(t))$. If $\rho = 1$ and $\beta = 0$ Theorem 2 of [5] makes $\{X(t)\}$ mixing and (a) follows from Theorem 3.2(b). Assume now that $\rho > 1$ and $\beta = 0$, which according to Lemma 4.2 may be achieved, if necessary, after a re-centering. Since by Remark 4.2 any skeleton chain $\{X(n\delta):n\geq 0\}$ assumes at least one $y_0 \neq 0$ of type I', we deduce that all points of $\{X(n\delta)\}$ must be of type I'. Indeed, by Theorem 4.1(b) $\alpha^k y_0$ is also a point of type I' for $\{X(n\delta)\}$. If we choose δ' with $\delta \neq \alpha^k y_0$ for all k, then by Theorem 3.1 $\{X(t), t \in U\}$ where $U = \{n\delta\} \cup \{n\delta'\}$ assumes the same points of type I' as $\{X(n\delta)\}$. Since δ' is at our disposal we conclude that $\{X(n\delta)\}$ assumes only points of type I'. Further, according to Remark 4.1

$F(x_2) - F(x_1) > 0$ implies $F(\rho^s x_2) - F(\rho^s x_1) > 0$ for any s, which makes F strictly increasing on it support. Remark 4.1 also implies that if $x \neq 0$ is a jump point for F then $\rho^s x$ is also a jump point for F, and s being arbitrary we would get an uncountable set of jump points, which is impossible. Thus, there are no jump points for F except may be for $x = 0$. Since F is continuous and strictly increasing on its support, an argument already used in the course of the proof of Lemma 2.5 yields that $\lim_{n\to\infty} \{X(t_n) \le x\}$ a.s. exists for any continuity point x of F and this is tantamount to a.s. convergence for $\{X(t_n)\}$. Since $\{X(t)\}$ was assumed right-continuous we conclude that $\{X(t)\}$ converges a.s. (see e.g. [22])

The case $\rho = 1$ and $\beta \neq 0$ may be treated in a similar way. Since $\{W=0\}$ is no longer invariant, 0 cannot be a jump point for F in this case.

5. <u>A criterion for a.s. convergence</u>. Theorem 4.1(b) asserts that under some conditions on ρ and β, convergence in distribution for $\{X(t)\}$ entails a.s. convergence. In many cases of interest it is rather difficult to derive convergence in distribution, such that a tractable criterion of this kind seems of interest. We shall derive here such a criterion assuming only tightness for $\{X(t)\}$ and a condition on the transition probability functions $\{P_s\}$ for $s \in (0, \delta)$ and some $\delta > 0$.

A random process $\{Y(t)\}$ will be said to be tight if any subsequence thereof contains another subsequence converging in distribution to a non-identically 0 random variable.

Further we shall consider the following conditions:

(A) Either $1 < \liminf_{t\to\infty} a(t+s)/a(t) \le \limsup_{t\to\infty} a(t+s)/a(t) < \infty$ or $0 < \liminf_{t\to\infty} a(t+s)/a(t) \le \limsup_{t\to\infty} a(t+s)/a(t) < 1$ for some $s > 0$.

(B) There exist $\delta > 0$ and $\rho \neq 1$ such that

$$\lim_{t\to\infty} P(|Y(t+s)/Y(t) - \rho^s| > \varepsilon \,|\, X(t) \neq 0) = 0$$

for any $\varepsilon > 0$ and $s \in (0,\delta)$.

(B1) There exist $\delta > 0$ and $\rho \neq 1$ such that

$$\lim_{t\to\infty} P(Y(t+s) \in (c(t)\rho^s(1-\varepsilon),\, c(t)\rho^s(1+\varepsilon)) \,|\, Y(t) = c(t)) = 1$$

for $c(t) = xa(t)$ with $x \in R$, $\varepsilon > 0$ and $s \in (0,\delta)$. The main result of this Section is the following:

Theorem 5.1. Suppose that $\{Y(t): t \in [0,\infty)\}$ is a non-negative SM Markov process with stationary transition probabilities, $X(t) = a(t)Y(t)$, where $\{a(t)\}$ are some constants that satisfy condition (A). Assume further that $\nu_t << \nu_s$ for $t > s$ where $\nu_t(\cdot) = P(Y(t) \in \cdot)$. Then the tightness of $\{X(t)\}$ in conjunction with condition (B1) is a necessary and sufficient condition for the existence of some constants $\{a'(t)\}$ with $\lim_{t\to\infty} a'(t+s)/a'(t) = \rho^s$ for all $s > 0$ such that $\{a'(t)Y(t)\}$ converges a.s. as $t \to \infty$ to a non-degenerate random variable X. If $F(x) = P(X \leq x)$ then supp F is either the real line or one of its half-lines, F is continuous except may be for $x = 0$, and strictly increasing on supp F.

Remark 5.1. In view of Theorem 4.1, condition (B) is necessary for a.s. convergence when $b(t) \equiv 0$. It may be shown that (B) entails (B1) if $\{X(t)\}$ is tight by reasoning in the manner of [8] (see also [7]).

In what follows we shall assume that the conditions of Theorem 5.1 are in force. We shall need the following two Lemmas:

Lemma 5.1. Suppose that for some left-unbounded intervals $\{I_t\}$ $\lim_{t\to\infty} \{Y(t) \in I_t\}$ a.s. exists. Then for any real s $\rho^s \lim_{t\to\infty} \{Y(t) \in I_t\} =$ $\lim_{t\to\infty} \{Y(t + s) \in I_t\}$ a.s. also exists.

Lemma 5.2. Suppose that $\{t_n\}$ is chosen such that $\{X(t_n)\}$ converges in distribution to a limit F as $t_n \to \infty$. Then F is non-degenerate, and there exists q with $F(0) < q < 1$ such that W_q is of type I.

We delay the proofs of the above Lemmas to explain now the idea of the proof.

Outline of the proof of Theorem 5.1. We shall confine ourselves to the case $b(t) = 0$ and $Y(t) \geq 0$. By Lemma 5.2 we know that there exists x such that $F(0) < P(W_q \geq x) < 1$. Since $\{Y(t)\}$ was assumed stochastically monotone, we deduce that

$$(5.1) \qquad \{W_q \geq x\} = \lim_{t\to\infty} \{Y(t) \in I_t\}$$

where I_t is either $(-\infty, x_t)$ or $(-\infty, x_t]$ for some numbers $\{x_t\}$. It will be shown that we may assume that $I_t = (-\infty, x_t]$ such that (5.1) and Lemma 5.1 imply that $\lim_{t\to\infty} \{Y(t + s) \leq x_t\}$ a.s. exists for all s. Since condition (B1) will turn out to lead to $\lim_{t\to\infty} x_{t+s}/x_t = \rho^s$ for some ρ with $\rho \neq 1$, we get

$$(5.2) \qquad \lim_{t\to\infty} \{Y(t + s) \leq x_t\} = \lim_{t\to\infty} \{Y(t) \leq \rho^s x_t\} \quad \text{a.s.}$$

As s in (5.2) is arbitrary, we conclude that $\lim_{t\to\infty} \{x_t^{-1} Y(t) \leq x\}$ a.s. exists for all x, which is tantamount to a.s. convergence for $\{x_t^{-1} Y(t)\}$.

Proof of Lemma 5.1. This lemma is a continuous time variant of Lemma 4.3.

Proof of Lemma 5.2. We shall assume that $1 < \liminf_{t\to\infty} a(t+s)/a(t) \leq \limsup_{t\to\infty} a(t+s)/a(t) < \infty$, as the other case satisfying condition (A) is reducible to this one by taking $1/Y(t)$ instead of $Y(t)$.

Choose x to be a continuity point of F and let $F(x) = q$. Then

$$(5.3) \qquad P(a(t_n)Y(t_n + s) \leq x) = \int P(X(t_n) \leq x \mid Y(0) = y)\nu_s(dy)$$

where $s > 0$. Taking the limit as $n \to \infty$ yields

$$(5.4) \qquad F_x^{(s)} = \int \hat{G}_y^{(0)}(q)\nu_s(dy) = E(\hat{G}_{Y_s}^{(0)})$$

where $F^{(s)}$ is the limit distribution of $\{a(t_n)Y(t_n + s)\}$ and $\hat{G}_y^{(t)}(q) = \lim_{n\to\infty} P(X(t_n) \leq x \mid Y(t) = y)$. Assume that F is degenerate. Then $\hat{G}_y^{(0)}(q) = 0$ or 1 a.s. with respect to ν_0 and ν_s, since $\nu_s \ll \nu_0$. By (5.4) $F^{(s)}(x) = F(x)$ and $\liminf_{t\to\infty} a(t+s)/a(t) > 1$ in conjunction with the tightness of $\{X(t)\}$ is contradicted. Thus F is non-degenerate. Suppose now that W_q is a.s. constant, i.e. $\hat{G}_y^{(0)}(q) = F(x)$ a.s. with respect to ν_0 and ν_s and as above we get $F^{(s)}(x) = F(x)$ for any $s > 0$, which is impossible. Thus W_q is not a.s. constant and therefore we may choose a point z, which is a continuity point of the distribution function of W_q, such that $0 < P(W_q \leq z) < 1$. Then by an already familiar argument we know that there exist some left-unbounded intervals $\{J_t\}$ such that $\lim_{t\to\infty} \{Y(t) \in J_t\} = \{W_q > z\}$ a.s. and if $P(W_q > z) > F(0)$, Lemma 2.2 concludes the proof. Assume therefore that $P(W_q > z) \leq F(0)$. According to Theorem 2.1 this situation corresponds to the case of W_q of type II with

$P(W_q = 1) > 0$ and $P(W_q = 1) + P(W_q = k_q) = 1$. Choose $z > k_q$. Since by Lemma 5.1 $\Theta^s\{W_q = 1\} = \lim_{t\to\infty} \{Y(t + s) \in J_t\}$ a.s., if we take into account the assumption $\liminf_{t\to\infty} a(t + s)/a(t) > 1$ we get $J_{t+s} \supseteq J_t$ for t large enough and $\Theta^s\{W_q=1\} \supseteq \{W_q=1\}$. However, we know from [6] that $P(W_q=1)<1$ entails $P(\Theta^s\{W_q=1\})<1$. Since W_q admits only two values with positive probability, Lemma 2.3(b) makes it impossible that $P(\Theta^s\{W_q=1\})>P(W_q=1)$. Thus $\{W_q=1\}$ is an invariant set, and since $\{W_q=k_q\}$ is its complementary set it must also be invariant. Therefore W_q is an invariant random variable. It follows that $E(G^{(0)}_{Y_s}(q)) = E(G^{(s)}_{Y_s}) = F(x)$ and (5.4) implies $F^{(s)}(x) = F(x)$ case which we considered before and turned out to be absurd.

Proof of Theorem 5.1. Step 1. We first show that if $\Lambda = \lim_{t\to\infty} \{Y(t) \in J_t\}$ a.s. where $F(0) < \lim_{t\to\infty} P(Y(t) \in J_t) < 1$ then $P(\Theta^s\Lambda) > P(\Lambda)$ for any $s > 0$. The existence of such Λ was ensured by Lemma 5.2. Recall that $\eta = \liminf_{t\to\infty} a(t+s)/a(t)>1$ and obviously $\liminf_{t\to\infty} a(t + ks)/a(t) \geq \eta^k$ for any $k > 0$. Notice further that if x_t is the right end-point of J_t, then, if necessary extracting a further subsequence of $\{t_n\}$, we may assume that $a(t_n) \sim cx_{t_n}$ where c is a positive constant. The above argument boils down to $P(\Theta^{ks}\Lambda) = \lim_{n\to\infty} P(Y(t_n + ks) \in J_{t_n}) \geq \limsup_{n\to\infty} P(X(t_n) \leq c^{-1}\eta^k) = F(c^{-1}\eta^k)$. As F is a proper distribution, we may find k large enough such that $F(c^{-1}\eta^k) > q$. Thus there is a k such that $P(\Theta^{ks}\Lambda) > P(\Lambda)$. However $\Theta^s\Lambda \supseteq \Lambda$ for any $s > 0$ as we have already noticed in the course of the proof of Lemma 5.2. This makes $P(\Theta^s\Lambda) > P(\Lambda)$ for $s > 0$ the only possibility and concludes the argument.

Step 2. We show now that if $\{s_n\}$ is a sequence of positive numbers with $\lim_{n\to\infty} s_n = 0$ then $\lim_{n\to\infty} \Theta^{s_n}\Lambda = \Lambda$ a.s. for any $\Lambda \in \mathcal{T}$, where $\mathcal{T}$ is the tail σ-field of $\{Y(t): t \in [0,\infty)\}$. Indeed, $\{Y(t)\}$ was assumed to be right-continuous, in which case it is known that $F_t = \lim_{n\to\infty} F_{t+s_n}$ (see e.g. [22]) where F_t is the σ-algebra generated by $\{Y(u): 0 < u \leq t\}$. Since $P(\Theta^{s_n}\Lambda|F_{t+s_n}) = P(\Lambda|F_t)$ for $t > 0$, we get $P(\Lambda'|F_t) = P(\Lambda|F_t)$ with $\Lambda' = \lim_{n\to\infty} \Theta^{s_n}\Lambda$ on letting $n\to\infty$. Because $\Theta^s\Lambda$ is decreasing in s and $\Theta^s\Lambda \supseteq \Lambda$ for $s > 0$ we conclude that $\Lambda = \Lambda'$ a.s.

Step 3. We shall next show that $\{Y(t)/x_t\}$ converges a.s. as $t\to\infty$ for some constants $\{x_t\}$. Indeed, choose $q \in (0,1)$ such that $F(x_0) = q$ for a continuity point x_0 of F. Then W_q must be of type I. Indeed, assume the contrary. Then by Lemmas 5.2 and 2.3, W_q must assume at least one positive value out of $P(W_q = 0)$ and $P(W_q = 1)$. Assume for definiteness that $P(W_q = 0) > 0$. Then $0 < F(x_0) < P(\Lambda_0) < 1$ where $\Lambda_0 = \{W_q > 0\} = \lim_{n\to\infty} \{X(t_n) \in J_{t_n}\}$ a.s. for some left-unbounded $\{J_{t_n}\}$. By Steps 1 and 2 we deduce that one may find Λ_0' with $\Lambda_0' = \Theta^{-s}\Lambda_0$ and $s > 0$ such that $F(x_0) < P(\Lambda') < P(\Lambda)$. By Lemma 5.1 we know that $\Lambda' = \lim_{n\to\infty} \{X(t_n) \in J_{t_n}'\}$ for some left-unbounded intervals $\{J_{t_n}\}$ and according to Lemma 2.2(b) $W_{q'}$, with $q' = P(\Lambda_0')$, is of type I, which contradicts Lemma 2.3(b) and proves that W_q is of type I. Thus x_0 is a point of type I and therefore there exist some left-unbounded intervals $\{I_t\}$ with right-end points $\{x_t\}$ such that $\Lambda = \lim_{t\to\infty} \{Y(t) \in I_t\}$ a.s. and $P(\Lambda) = F(x_0)$. It is further easy to see that

$$\lim_{n\to\infty} P(\{Y(t_n) \in I_{t_n}\} \Delta \{Y(t_n) \leq a(t_n)x_0\}) = 0 \tag{5.5}$$

It is obvious that $\lim_{n\to\infty} \{Y(t) \in I_t\} = \lim_{n\to\infty} \{Y(t_n) \in I_{t_n}\} = \lim_{n\to\infty} \{Y(t_n+s) \in I_{t_n+s}\}$ a.s.

and (5.5) leads to

(5.6) $$\lim_{n\to\infty} P(\{Y(t_n+s) \in I_{t_n+s}\} \,\Delta\, \{Y(t_n) \le a(t_n)x_0\}) = 0$$

On the other hand, condition (B1) implies

(5.7) $$\lim_{n\to\infty} P_s(a(t_n)(x_0 - \varepsilon),\ (-\infty, a(t_n)x_0\rho^s)) = 1$$

and

(5.8) $$\lim_{n\to\infty} P_s(a(t_n)(x_0 + \varepsilon),\ (-\infty, a(t_n)x_0\rho^s)) = 0$$

Stochastic monotonicity applied to (5.7) and (5.8) yields

(5.9) $$\lim_{n\to\infty} P_s(x, (-\infty, a(t_n)x_0\rho^s)) = 1$$

uniformly for $x < a(t_n)(x_0 - \varepsilon)$, and

(5.10) $$\lim_{n\to\infty} P_s(x, (-\infty, a(t_n)x_0\rho^s)) = 0$$

uniformly for $x > a(t_n)(x_0 + \varepsilon)$

Taking into account (5.9) and the continuity of F at x_0 we get

(5.11) $$F(x_0) = \lim_{n\to\infty} \int_{\{x \le a(t_n x_0)\}} P_s(x, (-\infty, a(t_n)x_0\rho^s)\nu_{t_n}(dx)$$

which is easily seen to be equivalent to

(5.12) $$\lim_{n\to\infty} P(Y(t_n) \le a(t_n)x_0) = \lim_{n\to\infty} P(\{Y(t_n + s) \le a(t_n)x_0\rho^s\} \cap \{Y(t_n) \le a(t_n x_0\})$$

where we have used the equality

$$\int_{\{x \le a(t_n)x_0\}} P_s(x, (-\infty, a(t_n)x_0\rho^s)\nu_{t_n}(dx) = P(\{Y(t_n + s) \le a(t_n)x_0\rho^s\} \cap \{Y(t_n) \le a(t_n)x_0\})$$

Proceeding in the same way as above, but using (5.1) instead of (5.9) we get

(5.13) $\lim_{n\to\infty} P(Y(t_n)>a(t_n)x_0) = \lim_{n\to\infty} P(\{Y(t_n+s)>a(t_n)x_0\rho^s\}\cap\{Y(t_n)>a(t_n)x_0\})$

It is now easy to see that (5.6), (5.12) and (5.13) yield

(5.14) $\lim_{n\to\infty} P(\{Y(t_n+s)\in I_{t_n+s}\}\ \Delta\ \{Y(t_n+s)\le a(t_n)x_0\rho^s\}) = 0$

Because x_0 was chosen to be an arbitrary continuity point of F, we get $\lim_{n\to\infty} x_{t_n+s}/x_{t_n} = \rho^{-s}$ and since $\{t_n\}$ was assumed to be an arbitrary sequence with $\lim_{n\to\infty} t_n = \infty$ such that $\{X(t_n)\}$ converges in distribution we get $\lim_{t\to\infty} x_{t+s}/x_t = \rho^{-s}$ for any $s\in(0,\delta)$. It is easy to see that the latter equality implies $\lim_{t\to\infty} x_{t+s}/x_t = \rho^{-s}$ for any real s. Recall that $\lim_{t\to\infty}\{Y(t+s)\in I_t\}$ a.s. exists for all s and the above considerations boil down to the existence of $\lim_{t\to\infty}\{Y(t)\le\rho^s x_t\}$ a.s. But ρ^s may take any value as s is at our disposal. It follows that $\{Y(t)/x_t\}$ converges a.s. to a limit X as $t\to\infty$, and X was shown to be non-degenerate by Lemma 5.2. Since Theorem 4.1 applies, its characterization of F carries over to this case.

Step 4. To prove that the conditions of Theorem 5.1 are necessary, notice first that tightness is an obvious prerequisite for convergence in distribution. Condition (B1) is obviously implied by the a.s. convergence of $\{Y(t)/x_t\}$ (as well as its equivalent form (B)) if we take i[illegible] account Theorem 4.1.

6. Applications. Diffusions. It has been noticed by several authors that diffusions are SM. Indeed, the birth-and-death process is SM(see e.g. [17]). Since by a result of Stone [26] any diffusion is a limit of birth-and-death processes

it follows that diffusions are SM. Next we shall give an a.s. convergence criterion for Markov processes assuming second moments that may be applied to diffusions. We need consider the following.

Condition (B2). There exist $\delta > 0$ and $\rho \neq 1$ such that

$$\lim_{t\to\infty} \frac{\mathrm{Var}(X(t+s)\,|\,X(t)=c(t))}{\min^2\,[\rho^s c(t)(1+\varepsilon)-E(X(t+s)\,|\,X(t)=c(t))],-[\rho^s c(t)(1-\varepsilon)-E(X(t+s)\,|\,X(t)=c(t))]\}} = 0$$

for $c(t) = xa(t)$ with $x \in R$, $\varepsilon > 0$ and $s \in (0,\delta)$

In what follows we shall write $u(t) \sim v(t)$ whenever $\lim_{t\to\infty} u(t)/v(t)=1$.

Theorem 6.1. Suppose that $\{Y(t):t \in [0,\infty)\}$ is a right-continuous SM Markov process with stationary transition probabilities, $\nu_t \ll \nu_s$ for $t > s$, $E(X(t)) \sim a\rho^t$ and $\mathrm{Var}\,(X(t)) \sim b\rho^{2t}$ for some constants a,b and ρ with $b > 0$ and $\rho \neq 1$, and that Condition (B2) holds. Then $\{Y(t)/\rho^t\}$ converges a.s. as $t \to \infty$ to a random variable X. If $F(x) = P(X \leq x)$ then supp F is either the real line or one of its half-lines, F is continuous except may be for $x = 0$, and strictly increasing on supp F.

Proof. We shall show that the conditions of Theorem 5.1 are verified. Indeed, by well-known properties for sequences of distribution functions (see e.g. [22]) any subsequence of $\{X(t)/\rho^t\}$ contains another subsequence whose limit distribution's variance equals b and is therefore non-degenerate. Thus tightness follows. It remains to show that (B2) implies (B1). Notice that

$$\begin{aligned} P(X \in (a,b)) &= P(X - E(X) \in (a - E(X),\, b - E(X)) \\ &\geq P(X \in (-c,c)) \end{aligned}$$

where $c = \min(b - E(X), -(a - E(X)))$. Specializing X, a and b to the quantities that appear in (B1) and applying the Chebyshev's inequality we get that (B2) implies (B1) and complete the proof. Examples of diffusions to which Theorem 6.1 applies include the Ornstein-Uhlenbeck processes (see e.g. [16]) and some diffusion processes that approximate Galton-Watson processes (see [11], [13] and [21]). In both cases $E(Y(t)|Y(0) = x) = xe^{\beta t}$ and Var $(Y(t)) \sim be^{2\beta t}$ with $b > 0$ and $\beta > 1$. Such results for Ornstein-Uhlenbeck processes are derived by using some heavy machinery developed for diffusion processes (see e.g. [24]). For branching diffusions some analytic tools are available (see [3]). However, even small perturbations in the transition probability functions of such processes may destroy the martingale properties on which their study is based, whereas the conditions of Theorem 6.1, being of the limit type, seem to be more robust to such changes.

Branching processes. We shall derive a limit theorem for a branching model in which the offspring of the individuals are no longer independent, but strictly stationary. Stochastic monotonicity methods seem to allow one to establish results where the classical proofs based on independence break down. We shall pare the assumption down to the bare essentials so that our conditions will be formulated in terms of properties that are used in establishing stochastic monotonicity results. We shall neither bother here with deriving assumptions on the process that entail such conditions nor with finding minimal conditions ensuring our results. A more comprehensive study of such processes will be taken up elsewhere. In [7] we studied SM branching models of [3], [13] and [25].

Suppose that $\{Z_t : t \in [0,\infty)\}$ is a Markov process such that

$$(6.1) \qquad Z_{t+u} = \sum_{i=1}^{Z_t} Z_{t,u}^{t,i} \text{ if } Z_t > 0 \text{ and } Z_{t+u} = 0 \text{ if } Z_t = 0$$

where $Z_{t,u}^{t,i}$ stands for the number of offspring at time t+u of the i-th of the Z_t individuals alive at time t.

In a Galton-Watson process, $\{Z_{t,u}^{t,i}\}$ are assumed i.i.d. and independent of Z_t.

Consider next the following conditions

C(1) The sequence $\{Z_{t,u}^{t,i};\ i = 1,2,\ldots\}$ is independent of Z_t and is distributed like the strictly stationary and ergodic process $\{\xi_i^{(u)};\ i = 1,2,\ldots\}$

C(2) $$P(\lim_{t\to\infty} Z_t = \infty) = 1 - P(\lim_{t\to\infty} Z_t = 0)$$

C(3) For any $\{x_u\}$ with $\lim_{u\to\infty} P(Z_{t,u}^{t,1} > x_u) \in (0,1)$ one gets

$$\lim_{u\to\infty} P(\{Z_{t,u}^{t,1} > x\} \cup \{Z_{t,u}^{t,2} > x_u\}) > \lim_{u\to\infty} P(\{Z_{t,u}^{t,1} > x_u\}) \tag{6.2}$$

Theorem 6.2. Suppose that $\{Z_t\}$ is a right-continuous process that satisfies conditions C(1), C(2) and C(3), and $E(Z_t) < \infty$. Then there exist some norming constants $\{c(t)\}$ with $\lim_{t\to\infty} c(t+s)/c(t) = e^{\alpha s}$ for some $\alpha > 1$ such that $\{Z(t)/c(t)\}$ converges a.s. to a random variable W. If $F(x) = P(W \le x)$, then F is continuous and strictly increasing on $(0,\infty)$.

Proof. Since $P_u(x,(-\infty,y]) = P(\sum_{i=1}^{x} Z_{t,u}^{t,i} \le y)$ we can easily see that increasing x means adding more non-negative variables to the sum, which of course decreases its probability of being smaller or equals x. Thus $\{X_t\}$ is SM. Notice now that (6.1) and C(1) lead to $E(Z_{t+u}) = E(Z_t)E(Z_u)$ whereas C(1) and C(2) yield $E(Z_t) > 1$. Thus there must exist $\alpha > 1$ such that $E(Z_t) = e^{\alpha t}$ for any $t > 0$.

Birkoff's ergodic theorem is easily seen to imply (B1) and also (B) in the form

$$(6.3)\qquad \lim_{t\to\infty} P(\,|Z_{t+s}/Z_t - e^{\alpha s}| > \varepsilon \,|\, Z_t \neq 0) = 0$$

for any $\varepsilon > 0$. If we define $a(t)$ such that $a^{-1}(t)$ is the γ-quantile of the distribution function of Z_t for $P(\lim_{t\to\infty} Z_t = 0) < \gamma < 1$ then by (6.3) we conclude that $1 < \liminf_{t\to\infty} a(t+s)/a(t) \le \limsup_{t\to\infty} a(t+s)/a(t) < \infty$, so that condition (A) holds. It is easy to see, by the way $\{a(t)\}$ were defined, that any weakly convergent subsequence of $\{a(t)Z_t\}$ must have a non-degenerate limit distribution F. To prove tightness for $\{a(t)Z_t\}$ we need to show that $F(\infty)=1$. Assume the contrary and choose $\{u_n\}$ with $\lim_{n\to\infty} u_n = \infty$ such that $\{a(t+u_n)Z_{t+u_n}\}$ converges in distribution to F and $\{a(t+u_n)Z^{t,1}_{t+u_n}\}$ converges in distribution to a limit G. Notice further that (6.1) leads to

$$(6.4)\qquad W \overset{\mathcal{D}}{=} \sum_{i=1}^{Z_t} W_{t,i}$$

where $\overset{\mathcal{D}}{=}$ means that W and $\sum_{i=1}^{Z_t} W_{t,i}$ have the same distribution, whereas W is distributed according to F and $\{W_{t,i}\}$ are distributed according to G. Further (6.4) leads to

$$(6.5)\qquad P(W = \infty) = \sum_{n=1}^{\infty} P\Big(\bigcup_{j=1}^{n} W_{t,i} = \infty\Big) P(Z_t = n)$$

Since $\{a(t)\}$ satisfies condition (A) we get $P(W = \infty) = P(W_{t,i} = \infty)$ and by C(3) the right-hand side of (6.5) would be larger than its left-hand side, which is absurd. Thus $P(W = \infty) = 0$ and the condition of Theorem 5.1 are checked.

As we mentioned before, the conditions of Theorem 6.2 may be relaxed.

Perturbation factors may be allowed in (6.1) whereas some kind of dependence for $\{Z^{t,i}_{t,u}\}$ on Z_t in the manner of [19] and [20] may supercede condition C(1).

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